A CONSTRUCTION OF THE UNIVERSAL COVER AS A FIBER BUNDLE

DANIEL A. RAMRAS

In these notes we present a construction of the universal cover of a path connected, locally path connected, and semi-locally simply connected space. This construction builds the universal cover as a fiber bundle, by gluing together various evenly covered pieces. This process is surprisingly straightforward, and requires one to think less about point-set issues than the standard construction presented in Hatcher [1, Section 1.3]. These notes are based on notes by Greg Brumfiel.

1. Fiber bundles and transition functions

We begin with the general definition of a fiber bundle. Note that covering spaces correspond precisely to fiber bundles in which the fibers are discrete spaces.

Definition 1.1. *A fiber bundle consists of a pair of spaces E, B and a map $E \xrightarrow{p} B$, subject to the following local triviality condition:*

For each point $b \in B$, there exists an open neighborhood $U \subset B$ (with $b \in B$), a space F_U, and a homeomorphism $\tau\colon p^{-1}(U) \xrightarrow{\cong} U \times F$ making the diagram

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\quad\tau\quad} & U \times F_U \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & U & \end{array}$$

commute (here π_1 is the projection onto the first factor).

We call E the total space, B the base space, and p the projection. We'll refer to the neighborhoods U appearing in the local triviality condition as locally trivial neighborhoods, and we call the maps τ local trivializations. Note that the space F_U is homeomorphic to each fiber $p^{-1}(x)$, $x \in U$, via the restriction of τ, so we refer to F_U as the fiber of p.

The local triviality condition implies that all fibers in a neighborhood of b are homeomorphic to the fiber over b itself. One can check that if B is connected, then all of the fibers $p^{-1}(b)$ are homeomorphic to one another (choose a point $b \in B$, and use the local triviality condition to show that $\{x \in B : p^{-1}(x) \cong p^{-1}(b)\}$ is both open and closed). Hence the spaces F_U above will all be homeomorphic to one another.

If the fibers F_U are discrete spaces, then this definition reduces to the definition of a covering space, because then $U \times p^{-1}(b)$ is a disjoint union of copies of U.

One should think of the local trivializations as giving "coordinates" for points in E, but these coordinates are valid only locally. Remember that in a covering space, two fibers can be identified with one another via path lifting. If we consider a simply connected neighborhood in the base, then within that neighborhood these identifications won't depend on the specific paths we choose. This is a way of obtaining local trivializations for covering spaces of (semi-)locally simply connected spaces, and forms the basic idea behind our construction of the universal cover.

1.1. **Transition functions.**

Transition functions offer a useful way to think about fiber bundles. Let $E \xrightarrow{p} B$ be a fiber bundle, and imagine that we have a point $b \in B$ that lies in two different locally trivial neighborhoods U_1 and U_2, with trivializations $\tau_i\colon p^{-1}(U_i) \to U_i \times F_i$ (respectively). We now have two different ways of trivializing $p^{-1}(U_1 \cap U_2)$, and we refer to the map

$$\tau_{21} = \tau_2\tau_1^{-1}\colon U_1 \cap U_2 \times F_1 \xrightarrow{\cong} U_1 \cap U_2 \times F_2 \qquad (1)$$

as a *transition function*: it tells us how to switch between the two local trivializations of our bundle, or in other words how to change coordinates.

For our purposes, the important thing will be to go in the other direction: starting with an open cover $\{U_i\}$ of a (connected) space B, and a collection of transition functions

$$\phi_{ji}\colon U_i \cap U_j \times F_i \xrightarrow{\cong} U_i \cap U_j \times F_j,$$

how can we build a fiber bundle $E \to B$ whose fibers are the (homeomorphic) spaces F_i?

The key observation is that in a fiber bundle, the transition maps always satisfy a relation known as the *cocycle condition.* Roughly speaking, this condition says that if we have three overlapping locally trivial neighborhoods U_i, U_j, and U_k (meaning that $U_i \cap U_j \cap U_k \neq \emptyset$) with local trivializations τ_i, then changing from τ_i coordinates to τ_j coordinates, and then to τ_k coordinates, is actually the same as changing directly from τ_i coordinates to τ_k coordinates. This idea is summed up by the following equation:

$$\tau_{kj} \circ \tau_{ji} = \tau_{ki},$$

which is immediate from Definition (1). Note that we've indexed the transition functions in such a way that the cocycle condition looks like a cancellation between the two adjacent j's.

The following result let's us go in the other direction: we can construct a fiber bundle out of locally trivial pieces, together with transition functions satisfying the cocycle condition.

Proposition 1.2. *Let B be a space with an open covering $\{U_i\}_{i\in I}$, and let $\{F_i\}_{i\in I}$ be a collection of spaces. Consider a collection of continuous maps*

$$\tau_{ji}\colon U_i \cap U_j \times F_i \xrightarrow{\cong} U_i \cap U_j \times F_j,$$

one for each pair $(i,j) \in I \times I$ with $U_i \cap U_j \neq \emptyset$, and assume these functions satisfy $\pi_1 \tau_{ji} = \pi_1$ (where π_1 denotes the projection to $U_i \cap U_j$) and also satisfy the cocycle condition

$$\tau_{kj}\tau_{ji} = \tau_{ki}$$

whenever $U_i \cap U_j \cap U_k \neq \emptyset$.

Define a relation $\sim$ of the set $\coprod_i U_i \times F_i$ by

$$(u,f)_i \sim (u, \tau_{ji} f)_j$$

for every $u \in U_i \cap U_j$ and every $f \in F_i$, where the subscripts on the ordered pairs mean that the first is considered as an element of $U_i \times F_i$ and the other as an element of $U_j \times F_j$. Then $\sim$ is an equivalence relation, and the quotient space

$$E = \left(\coprod_i U_i \times F_i \right) / \sim$$

is a fiber bundle over B with projection map $p\left([(u,f)]\right) = u$.

Proof. First, note that taking $i = j$ in the cocycle condition tells us that $\tau_{ii}\colon U_i \times F_i \to U_i \times F_i$ is the identity map, and similarly setting $k = i$ tells us that τ_{ji} and τ_{ij} are inverses of one another. In particular, each τ_{ij} is a homeomorphism. With these observations, the cocycle condition translates immediately to the statement that $\sim$ is transitive, reflexive, and symmetric.

To check that $E \xrightarrow{p} B$ is a fiber bundle, we will show that the map

$$U_i \times F_i \hookrightarrow \coprod_i U_i \times F_i \longrightarrow \left(\coprod_i U_i \times F_i \right) / \sim \; = E,$$

which we denote α_i, gives rise to a local trivialization over U_i. First, note that the image of this map is $p^{-1}(U_i)$, because any equivalence class of the form $[(u,f)_j]$, with $u \in U_i \cap U_j$, has a representative $\tau_{ij}(u,f)_j \in U_i \times F_i$. We will show that α_i is a homeomorphism onto its image; the desired local trivialization is then $\tau_i = \alpha_i^{-1}$. It suffices to check that α_i is an open map. Say $V \subset U_i \times F_i$ is open. By definition of the quotient topology on E, we need to show that

$$\{(u,f)_j \in U_j \times F_j \,:\, (u,f)_j \sim (u',f')_i \in V\}$$

is open in $U_j \times F_j$. But this set is empty when $U_i \cap U_j = \emptyset$, and otherwise it is precisely $\tau_{ji}(V)$, which is open because τ_{ji} is a homeomorphism (note here that the relation induced by the τ_{ji} is already transitive!). □

Exercise 1.3. *Show that the transition functions associated to the local trivializations of E that we just built are none other than the maps τ_{ji} we started with.*

2. The universal cover

With the preliminaries out of the way, let's construct the universal cover. Let X be a path connected, locally path connected, semi-locally simply connected space. We want to build a path connected covering space $\widetilde{X} \xrightarrow{p} X$ with $\pi_1(\widetilde{X}) = 0$.

By assumption, we can cover X by open neighborhoods U_i such that

- each U_i is path connected, and
- the maps $\pi_1(U_i, u_i) \to \pi_1(X, x_0)$ are trivial.

Note that the second condition is true for one choice of basepoints if and only if it's true for all choices of basepoints: either way, the condition amounts to saying that every loop in U_i is nullhomotopic in X. Said another way,

$$\begin{gathered}\text{Any two paths in } U_i \text{ with the same endpoints are homotopic in } X, \\ \text{through a homotopy fixing the basepoints.}\end{gathered} \tag{2}$$

This is the form of the condition that we will use below. These will be the evenly covered (i.e. locally trivial) neighborhoods for the universal cover.

We need to decide how to represent the fibers of $\widetilde{X}$. Just as in Hatcher's construction, we'll think of the fiber over $x \in X$ as the collection of (based) homotopy classes of paths from x_0 to x. Here $x_0 \in X$ is a basepoint that will be fixed throughout our discussion. The idea behind this is simple: if we had the universal cover $\widetilde{X}$ in front of us, we could fix a point $\widetilde{x_0}$ in the fiber over x_0 and connect each point in the fiber over x to x_0. These paths, projected down into X, would give representatives for all the based homotopy classes of paths between x and x_0, because each such path would lift to a path in $\widetilde{X}$ ending somewhere in the fiber over x. So we will choose points $u_i \in U_i$ define F_i to be the set of all based homotopy classes of paths from x_0 to u_i, with the *discrete* topology. (Remember that fiber bundles with discrete fibers are covering spaces.) Condition (2) tells us that within U_i we can canonically identify F_i with the set of based homotopy classes of paths from x_0 to any other point $u \in U_i$. This corresponds to the discussion in the paragraph just before Section 1.1.

Next, we need to construct transition functions

$$\tau_{ji}\colon U_i \cap U_j \times F_i \longrightarrow U_i \cap U_j \times F_j.$$

Given $(x, [q]) \in U_i \times F_i$, we need to define $\tau_{ji}(x, [q]) = (u, [q'])$ where q' is a path from x_0 to u_j. To figure out what q' should be, think of $(x, [q]) \in U_i \times F_i$ as a homotopy class of paths from x_0 to x as follows: choose a path α in U_i from u_i to x, and now $q\alpha$ is a path to x. Note that $[q\alpha]$ (the based homotopy class of this path in the full space X) is independent of α, by Property (2). We choose q' so that $(x, [q']) \in U \times F_j$ corresponds to $(x, [q\alpha])$ under this same convention: choose any path β in U_j with $\beta(0) = u_j$ and $\beta(1) = x$, and set $q' = q\alpha\overline{\beta}$. Again, this does not depend on the choice of α or β (so long as these paths stay inside U_i and U_j, respectively).

To summarize, we have defined

$$\tau_{ji}(x, [q])_i = (x, [q\alpha\overline{\beta}])_j,$$

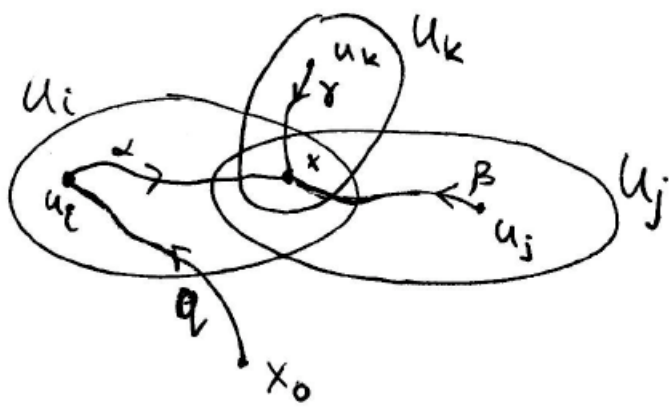
where α and β are paths in U_i and U_j (respectively) from u_i and to u_j to x.

Theorem 2.1. *There is a fiber bundle $\widetilde{X} \xrightarrow{p} X$ associated to the above data, and it is a simply connected covering space of X.*

Proof. We need to check the cocycle condition, in order to see that $\widetilde{X}$ is well-defined. This is simple enough: if α, β, and γ are paths in U_i, U_j, and U_k from u_i, u_j, and u_k to $x \in U_i \cap U_j \cap U_k$ (respectively), and q is a path from x_0 to u_i, then

$$\tau_{kj}\tau_{ji}(x, [q])_i = \tau_{kj}(x, [q\alpha\overline{\beta}])_j = (x, [q\alpha\overline{\beta}\beta\overline{\gamma}])_k = (x, [q\alpha\overline{\gamma}])_k = \tau_{ki}(x, [q])_i.$$

This is illustrated by the following picture:



By Proposition 1.2 we now have a fiber bundle $\widetilde{X} \xrightarrow{p} X$, and since the fibers are discrete this is actually a covering space. We need to show that $\widetilde{X}$ is path connected and that $\pi_1(\widetilde{X}) = 0$. Since q is a covering map, we know that q_* is injective on fundamental groups, with image the set of (homotopy classes of) loops in X whose lifts to $\widetilde{X}$ are loops. So to prove that $\widetilde{X}$ is simply connected, it will suffice to show if γ is a loop in X that lifts to a loop in $\widetilde{X}$, then γ is null-homotopic in X.

We will use one of the points $u_i \in U_i$ as our basepoint, and we'll write it as $x_0 \in U_0$. We will show that if q is a path in X from x_0 to $x \in U_i$, then the lift of q to $\widetilde{X}$ starting at $\widetilde{x_0} = [(x_0, [c_{x_0}])]$ ends at $[(x, [q\overline{\alpha}])_i]$, where (as above) α is any path in U_i from u_i to x. (Note that under the convention set up above, we can think of this point as $(x, [q])$.)

Assuming this lifting property for the moment, we check that $\widetilde{X}$ is path connected. If we choose any point $[(x, [q])_i] \in \widetilde{X}$, then the lift of q starting at $\widetilde{x_0}$ ends at $[(x, [q])_i]$. So this point is connected to $\widetilde{x_0}$ by a path in $\widetilde{X}$. The proof that $\widetilde{X}$ is simply connected is similar: if γ is a loop in X (at x_0), then its lift it to $\widetilde{X}$, starting at $\widetilde{x_0}$, ends at $[(x_0, [\gamma])]$ (we may choose the path α to be constant). This will be a loop if and only if $[\gamma] = [c_{x_0}]$, so only null-homotopic loops lift to loops.

To examine how a path γ in X lift to $\widetilde{X}$, we use the Lebesgue Lemma: there exists a decomposition of γ into a sequence of paths $\gamma_0, \gamma_1, \cdots, \gamma_k$ such that γ_i lies entirely inside of some U_i (since we are really just working with paths up to homotopy, we can reparametrize the γ_i so that they are defined on $[0,1]$). We may assume that U_0 is the piece of our cover containing x_0 that we chose above. Choose paths α_i in U_i from u_i to $\gamma_i(1)$.

We will prove, by induction on k, that the lift of γ ends at

$$[(\gamma_k(1), [\gamma_1\gamma_2\cdots\gamma_k\overline{\alpha_k}])_k].$$

When $k = 1$, the path γ stays inside the locally trivial neighborhood U_0 and hence its lift stays inside $p^{-1}(U_0) \cong U_0 \times F_0$ and hence must be constant in the second coordinate. In other words, the lifted path is simply $t \mapsto [(\gamma(t), [c_{x_0}])_0]$, and ends at $[(\gamma(1), [c_{x_0}])]$. Since loops in U_0 are null-homotopic in X, we have $[\gamma_0\overline{\alpha_0}] = [c_{x_0}]$, as desired.

For the induction step, we assume that the lift of $\gamma_1\cdots\gamma_{k-1}$ ends at $[(\gamma_{k-1}(1), [\gamma_1\cdots\gamma_{k-1}\overline{\alpha_{k-1}}])_{k-1}]$. Applying the transition function $\tau_{k\,k-1}$, this point is the same as

$$\begin{aligned}&[(\gamma_{k-1}(1), [\gamma_1\cdots\gamma_{k-1}\overline{\alpha_{k-1}}\alpha_{k-1}\gamma_k\overline{\alpha_k}])_k]\\&\quad= [(\gamma_{k-1}(1), [\gamma_1\cdots\gamma_{k-1}\gamma_k\overline{\alpha_k}])_k] = [(\gamma_{k-1}(1), [\gamma\overline{\alpha_k}])_k]\end{aligned}$$

as desired. (Note here that we are free to use the path $\gamma_k\overline{\alpha_k}$ from $\gamma_{k-1}(1)$ to u_k in computing the transition function $\tau_{k\,k-1}$.) □

3. PROBLEMS

Exercise 3.1. *Let G be a group, and consider a fiber bundle built using Proposition 1.2, in which $F_i = G$ for each i and the transition functions $\tau_{ji}\colon U_i \cap U_j \times G \to U_i \cap U_j \times G$ are* equivariant, *in the sense that $\tau_{ji}(x, gh) = g \cdot \tau_{ji}(x, h)$. Here G acts on $U_j \times G$ by $g \cdot (x, h) = (x, gh)$. Prove that there is a well-defined left action of G on the fiber bundle E associated to this data. (In this problem, you may assume that G has the discrete topology if you like, but that won't really be important. Any topological group would do.)*

Fiber bundles of the sort constructed in Exercise 3.1 are called *principal* bundles, or more specifically principal G–bundles.

Exercise 3.2. *In this exercise, we'll modify the construction of the universal cover so as to make it a principal $\pi_1(X, x_0)$–bundle. In our construction above, we viewed the fiber of $\widetilde{X}$ over $x \in X$ in terms of paths from x_0 to x. Now we'll make some choices so that each fiber becomes a copy of $\pi_1(X, x_0)$. Let $\{U_i\}_{i\in I}$ be a covering of X consisting of path connected open sets such that $\pi_1 U_i \to \pi_1 X$ is trivial. Choose points $u_i \in U_i$ and paths q_i from x_0 to u_i. Then we have maps $F_i \to \pi_1(X, x_0)$ sending $[q]$ to $[q\overline{q_i}]$.*

(1) *Show that these maps are bijections, so we can use them to identify* F_i *with* $\pi_1(X, x_0)$.
(2) *After making these identifications, the transition maps* τ_{ji} *used above can be identified with maps*

$$U_i \cap U_j \times \pi_1(X, x_0) \to U_i \cap U_j \times \pi_1(X, x_0).$$

Calculate these maps, and show that they satisfy the equivariance condition from Problem 3.1.
(3) *Problem 3.1 now tells us that* $\pi_1(X, x_0)$ *acts on* $\widetilde{X}$ *from the left. Show that this action corresponds to the action of* $\mathrm{Aut}(\widetilde{X})$ *on* $\widetilde{X}$*, under the isomorphism* $\pi_1(X, x_0) \xrightarrow{\cong} \mathrm{Aut}(\widetilde{X})$ *constructed in Hatcher* [1, Proposition 1.39] *(note that this isomorphism depends on a choice of basepoint in* $\widetilde{X}$*: you'll need to choose the basepoint* $[(x_0, c_{x_0})]$*).*
(4) *Show that after identifying the fiber* $p^{-1}(x_0)$ *with* $\pi_1(X, x_0)$ *by sending* $[\gamma] \in \pi_1(X, x_0)$ *to* $\widetilde{\gamma}_{\widetilde{x_0}}(1)$*, the above left action (restricted to this fiber) corresponds to* left multiplication *in* $\pi_1(X, x_0)$.

Exercise 3.3. *Let* $\widetilde{X} \xrightarrow{p} X$ *be the universal cover. The fundamental group* $\pi_1(X, x_0)$ *acts on* $\widetilde{X}$ *via deck transformations, as in the previous problem. But it also acts on the fiber* $p^{-1}(x_0)$ *via path lifting: given* $\widetilde{x}$ *in this fiber and* $[\gamma] \in \pi_1(X, x_0)$*, we can lift* γ *to a path* $\widetilde{\gamma}_{\widetilde{x}}$ *starting at* $\widetilde{x}$*, and we set* $\widetilde{x} \cdot [\gamma] = \widetilde{\gamma}_{\widetilde{x}}(1)$.
(1) *Prove that this defines a* right *action of* $\pi_1 X$ *on the fiber* $p^{-1}(x_0)$.
(2) *Show that after identifying the fiber* $p^{-1}(x_0)$ *with* $\pi_1(X, x_0)$ *by sending* $[\gamma] \in \pi_1(X, x_0)$ *to* $\widetilde{\gamma}_{\widetilde{x_0}}(1)$*, the above right action corresponds to* right multiplication *in* $\pi_1(X, x_0)$.

The next two exercises are meant to emphasize the fact that the right and left actions of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ are really very different actions.

Exercise 3.4. *Check that these actions agree (in the sense that* $[\gamma]\cdot\widetilde{x} = \widetilde{x}\cdot[\gamma]$*) if and only if* $\pi_1(X, x_0)$ *is an abelian group.*

Exercise 3.5. *Any left action* $G \times S \to S$ *of a group* G *on a set* S *can be converted into a right action by setting* $s \cdot g = g^{-1}s$*. Consider what happens when we convert the left action of* $\pi_1(\widetilde{X}, x_0)$ *on* $p^{-1}(x_0)$ *(from Exercise 3.2) into a right action. Show that this action agrees with the right action from Exercise 3.3 if and only if every element in* $\pi_1(X, x_0)$ *has order two. (This exercise could equally well have been phrased in terms of converting the action in Exercise 3.3 into a left action.)*

References

[1] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003-8001, USA
E-mail address: ramras@nmsu.edu