Slepian vectors on Boolean cubes
ICCHA7

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Outline

- Motivation: Extension of time and bandlimiting to graphs
- Techniques to compute eigenvectors
Time and band limiting on $\mathbb{R}$

- Bandlimiting: $(P_\Omega f)(x) = \left( \hat{f} \mathbb{1}_{[-\Omega/2, \Omega/2]} \right)^\vee (x)$  $P = P_1$
- Time limiting: $(Q_T f)(x) = \mathbb{1}_{[-T, T]}(x) f(x)$  $Q = Q_1$
- Time and band limiting: $P_\Omega Q_T$
$P_\Omega Q$ commutes with prolate differential operator

$$\mathcal{P}_\Omega : \frac{d}{dt}(t^2 - 1) \frac{d}{dt} + \pi^2 \Omega^2 t^2$$

Eigenfunctions: Prolate Spheroidal Wave Functions

Accurate methods to compute prolates from $\mathcal{P}_\Omega$
Figure: $\varphi_n, n = 0, 3, 10, c = \pi T \Omega / 2 = 5$
Properties of PSWFs

- Optimal time–concentration over all \( f \in \text{PW}_\Omega \)
- ONB for \( \text{PW}_\Omega \), . . . , orthogonal & complete in \( L^2[-1, 1] \)
- Discrete and finite dimensional analogues: Slepian sequences
Theorem (HJ Landau and H Widom, 1980)

For any $0 < \alpha < 1$ the number $N(\alpha)$ of eigenvalues of $P_{\Omega} Q_T$ greater than $\alpha$ satisfies (as $T \to \infty$)

$$N(\alpha) = 2\Omega T + \left( \frac{1}{\pi^2} \log \frac{1 - \alpha}{\alpha} \right) \log \left( 2\Omega T \right) + o(\log 2\Omega T).$$
Figure: Eigenvalues of $P_{\Omega}Q$ for $2\Omega = 10$ (blue), $2\Omega = 20$ (red) and $2\Omega = 50$ (yellow)
Boolean hypercube

- $\mathcal{B}_N = \{0, 1\}^N \sim \mathbb{Z}_2^N$
- Unweighted metric graph: vertices $\leftrightarrow S \subset \{1, \ldots, N\}$, Hamming distance
- (Unnormalized) graph Laplacian: $L$
- Eigenvalues: $0, 2, \ldots, 2N$ multiplicity $\binom{N}{K}$
- Eigenvectors: Hadamard vectors $H_S(R) = (-1)^{|R \cap S|}$; $LH_S = 2|S|H_S$
- Normalized: $\bar{H} = H/2^{N/2}$
Slepian questions for $\mathcal{B}_N$

- What are the analogues of time limiting and bandlimiting?
- What results on $\mathbb{R}, \mathbb{T} \leftrightarrow \mathbb{Z}, \mathbb{Z}_N$ hold on $\mathcal{B}_N$?
- Why $\mathcal{B}_N$ …
- Simple group structure/Fourier transform and homogeneity
- Nbhds grow $\Rightarrow$ concentration at *low frequencies* is a problem
Analogues of time and band limiting operators and prolate operator

- Let $D^2 = L$ (no minus sign)
- $D = HTH$, $T$ diagonal, $T_{SS} = \sqrt{2|S|}$
- Prolate analogue: BDO: $D(\alpha I - T^2)D + \beta T^2$
- Space limiting: $Q_{SS} = 1$, $|S| \leq K$, $Q_{SS} = 0$ else
- Band limiting: $P_K = \bar{H}Q_K\bar{H}$
Figure: $64 \times 64$ principal minor of BDO matrix conjugated by $\tilde{H}$ (left) and corresponding minor of $PQP$ for $(N, K) = (7, 3)$. Entries are indexed in binary lexicographic order.
Figure: Eigenvalues of $PQP$ (solid) and of $PBDO$ for $N = 7$ and $K = 3$. For $PBDO$ we plot $(\mu_n - \mu_1)/(\mu(64) - \mu_1)$ (with counter starting from one)
Figure: Eigenvectors (one per eigenvalue level) for \((N, K) = (7, 3)\). The first, fourth, and sixth vectors (and ninth, not shown) come from eigenvalues of multiplicity one, and are radial: constant on Hamming spheres.
Proposition

If $\alpha = 2\sqrt{K(K + 1)}$ then $P_K$ commutes with $D(\alpha l - T^2)D + \alpha T^2$

Proposition

The commutator of $D(\alpha l - T^2)D + \alpha T^2$ and $Q = Q_K$ is $[Q, \bar{H}[TlT, \bar{H}]] \neq 0$ (but small)
Commentary

- No apparent analogue of $2\Omega T$ theorem. Why?
- Eigenvector properties, including spectral accumulation (double orthogonality, etc)
- Extensions to other graphs
Estimating eigenvectors of $PQP$

- $D(\alpha I - T^2)D + \alpha T^2$ almost commutes with $Q_K P_K Q_K$, $\alpha = 2\sqrt{K(K+1)}$
- Eigenvectors of $PQP$ can be estimated starting from eigenvectors of $D(\alpha I - T^2)D + \alpha T^2$
Figure: $\log_{10}$ of norms of differences between successive iterates under $PQP$ with orthogonal projection from eigenvectors of BDO, $N = 7$, $K = 3$. Horizontal axis labels initial ranking of eigenvectors of BDO by the norms of their images under $PQP$. Power iteration until iteration limit is reached or iteration error is small.
Figure: $\log_{10}$ of norms of differences between successive normalized iterates of $PQP$ starting from eigenvectors of BDO (left) and random input vectors (right), $N = 9$, $K_1 = 4$, $K_2 = 3$ based on power iteration with orthogonal projection.

(a) $N = 9$, $K_1 = 4$, $K_2 = 3$

(b) $N = 9$, $K_1 = 4$, $K_2 = 3$, random input
Figure: Eigenvalue of PQP (red) and $\ell^2$ error between starting eigenvector of BDO and iteration eigenvector of PQP (blue), $(N, K) = (10, 3)$
Organize eigenvectors within eigenspaces (*natural* basis for eigenspace?)

Dimension of eigenspace

Behavior as $N \to \infty$: commute in limit? Commutator values $\leftrightarrow$ hypergeometric $\text{}_2F_1$

Other graphs