ABELIAN SUBALGEBRAS AND THE JORDAN STRUCTURE OF A VON NEUMANN ALGEBRA

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Abstract. For von Neumann algebras $\mathcal{M}, \mathcal{N}$ not isomorphic to $\mathbb{C} \oplus \mathbb{C}$ and without type $I_2$ summands, we show that for an order-isomorphism $f : \text{AbSub} \mathcal{M} \to \text{AbSub} \mathcal{N}$ between the posets of abelian von Neumann subalgebras of $\mathcal{M}$ and $\mathcal{N}$, there is a unique Jordan $\ast$-isomorphism $g : \mathcal{M} \to \mathcal{N}$ with the image $g[S]$ equal to $f(S)$ for each abelian von Neumann subalgebra $S$ of $\mathcal{M}$. The converse also holds. This shows the Jordan structure of a von Neumann algebra not isomorphic to $\mathbb{C} \oplus \mathbb{C}$ and without type $I_2$ summands is determined by the poset of its abelian subalgebras, and has implications in recent approaches to foundational issues in quantum mechanics.

1. Introduction

We consider the question: given a von Neumann algebra $\mathcal{M}$, how much information about $\mathcal{M}$ is encoded in the order structure of its collection of unital abelian von Neumann subalgebras? The set $\text{AbSub} \mathcal{M}$ of such subalgebras, partially ordered by set inclusion, becomes a complete meet semilattice in which every subset that is closed under finite joins has a join. The task is to reconstruct algebraic information about the algebra $\mathcal{M}$ from the order-theoretic structure of $\text{AbSub} \mathcal{M}$. More generally, we are interested in the interplay between these two levels of algebraic structure.

When $\mathcal{M}$ is abelian, the projection lattice $\text{Proj} \mathcal{M}$ forms a complete Boolean algebra, and one can show that the poset $\text{AbSub} \mathcal{M}$ is isomorphic to the lattice of complete Boolean subalgebras of $\text{Proj} \mathcal{M}$. Modifying a result of Sachs [26] that every Boolean algebra is determined by its lattice of all subalgebras, to show each

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complete Boolean algebra is determined by its lattice of complete subalgebras, one can then obtain that \( \text{Proj} \, \mathcal{M} \) is determined by \( \text{AbSub} \, \mathcal{M} \). That \( \mathcal{M} \) is determined by \( \text{Proj} \, \mathcal{M} \) is a consequence of the spectral theorem.

For a non-abelian von Neumann algebra, the situation is more complicated. Reconstruction of the non-commutative product in \( \mathcal{M} \) will not generally be possible as there are non-isomorphic von Neumann algebras having the same Jordan product, hence exactly the same posets of unital abelian subalgebras. However, we will show that the order structure of \( \text{AbSub} \, \mathcal{M} \) does determine \( \mathcal{M} \) as a Jordan algebra up to (Jordan) \( \ast \)-isomorphism. This means that the poset \( \text{AbSub} \, \mathcal{M} \) encodes a substantial amount of algebraic information about \( \mathcal{M} \). The proof goes along the same lines as the abelian case, using a result of [16] that an orthomodular lattice is determined by its poset of Boolean subalgebras. In fact, our result is somewhat stronger than we described.

**Theorem.** Suppose \( \mathcal{M}, \mathcal{N} \) are von Neumann algebras without type \( I_2 \) summands and \( f : \text{AbSub} \, \mathcal{M} \to \text{AbSub} \, \mathcal{N} \) is an order-isomorphism. Then there is a unique Jordan \( \ast \)-isomorphism \( F : \mathcal{M} \to \mathcal{N} \) with \( f(S) \) equal to the image.

This result is particularly interesting with respect to the so-called topos approach to the formulation of physical theories [6, 7, 8, 9, 10, 19], where a mathematical reformulation of algebraic quantum theory is suggested. For a von Neumann algebra \( \mathcal{M} \), one considers the poset \( \text{AbSub} \, \mathcal{M} \) of its abelian subalgebras and the topos of presheaves over this poset. The idea is that each abelian subalgebra represents a ‘classical perspective’ on the quantum system. By taking all classical perspectives together, one obtains a complete picture of the quantum system. Mathematically, this corresponds to considering the poset \( \text{AbSub} \, \mathcal{M} \) and presheaves over it. These presheaves form the topos associated with the quantum system. The so-called spectral presheaf \( \Sigma^\mathcal{M} \), whose components are the Gelfand spectra of the abelian von Neumann subalgebras of \( \mathcal{M} \), plays a key role in the topos approach. Physically, the spectral presheaf is interpreted as a generalized state space for the quantum system described by the algebra \( \mathcal{M} \). Mathematically, \( \Sigma^\mathcal{M} \) is a kind of spectrum of the non-abelian von Neumann algebra \( \mathcal{M} \). It becomes clear that, from the perspective of the topos approach, it is very relevant to see how much information about the algebra \( \mathcal{M} \) can be extracted from the poset \( \text{AbSub} \, \mathcal{M} \).

Since the appearance of the draft of this manuscript on ArXiv [5], several related manuscripts and papers have arisen. In [14] a related task is undertaken for the poset of abelian subalgebras of a \( C^\ast \)-algebra, and in [15] the matter is considered from the viewpoint of associative subalgebras of a Jordan algebra. In [4] applications to the topos approach to physical theories are considered further.
In particular, it is shown that if $M, N$ are von Neumann algebras with no direct summands of type $I_2$, then there is a Jordan $\ast$-isomorphism $F : M \to N$ if and only there is an isomorphism $\Phi : \Sigma^N \to \Sigma^M$ between their spectral presheaves in the opposite direction.

2. Preliminaries

For a complex Hilbert space $H$, let $\mathcal{B}(H)$ be the $C^\ast$-algebra of all bounded operators on $H$. For a subset $\mathcal{S} \subseteq \mathcal{B}(H)$, the commutant $\mathcal{S}'$ is the set of all elements of $\mathcal{B}(H)$ that commute with each member of $\mathcal{S}$. A von Neumann algebra is a subset $\mathcal{M} \subseteq \mathcal{B}(H)$ with $\mathcal{M} = \mathcal{M}'$. For a von Neumann algebra $\mathcal{M}$, we use $\text{Proj} \mathcal{M}$ for the set of projections in $\mathcal{M}$. The following well-known result [22, pg. 69] will be used repeatedly.

**Proposition 2.1.** For $\mathcal{M}$ a von Neumann algebra, $\mathcal{M} = (\text{Proj} \mathcal{M})''$.

For any von Neumann algebra $\mathcal{M}$ the projections $\text{Proj} \mathcal{M}$ form a complete orthomodular lattice (abbreviated: OML). Our primary interest lies in subalgebras of von Neumann algebras, subalgebras of their projection lattices, and relationships between these and the original von Neumann algebra. We require several definitions.

**Definition 2.2.** A von Neumann subalgebra of a von Neumann algebra $\mathcal{M}$ is a subset $\mathcal{S} \subseteq \mathcal{M}$ that is itself a von Neumann algebra.

We will only consider von Neumann subalgebras $\mathcal{S} \subseteq \mathcal{M}$ such that the unit elements in $\mathcal{S}$ and $\mathcal{N}$ coincide. (In particular, we will not consider subalgebras of the form $\hat{P}M\hat{P}$ for a non-trivial projection $\hat{P} \in \mathcal{M}$.) We remark that being a von Neumann subalgebra is equivalent to being a unital $C^\ast$-subalgebra that is closed in the $\sigma$-weak topology, equivalent to being a unital $C^\ast$-subalgebra that is closed under monotone joins [1, pg. 101–110].

**Definition 2.3.** For a von Neumann algebra $\mathcal{M}$, we let $\text{Sub} \mathcal{M}$ be the set of all von Neumann subalgebras of $\mathcal{M}$ ordered by set inclusion; $\text{AbSub} \mathcal{M}$ be the set of abelian von Neumann subalgebras of $\mathcal{M}$ ordered by set inclusion; and $\text{FabSub} \mathcal{M}$ be the set of all abelian subalgebras of $\mathcal{M}$ that contain only finitely many projections, ordered by set inclusion.

We note that $\text{Sub} \mathcal{M}$ is a complete lattice, with meets given by intersections. The join of a family $(\mathcal{S}_i)_{i \in I}$ of subalgebras is the weak closure of the algebra generated by the algebras $\mathcal{S}_i$, $i \in I$. Analogously, $\text{AbSub} \mathcal{M}$ is a complete meet semilattice where every subset that is closed under finite joins has a join, and
$F\text{AbSub} \mathcal{M}$ is a complete meet semilattice where every meet is essentially finite. Yet, neither $\text{AbSub} \mathcal{M}$ nor $F\text{AbSub} \mathcal{M}$ have a top element if $\mathcal{M}$ is non-abelian, so empty meets do not exist in these posets.

**Definition 2.4.** For an oml $L$, we let $\text{Sub} L$ be the set of all subalgebras of $L$; $B\text{Sub} L$ be the set of Boolean subalgebras of $L$, and $F\text{BSub} L$ be the set of finite Boolean subalgebras of $L$, all partially ordered by set inclusion. If $L$ is complete we let $C\text{Sub} L$ be the set of complete subalgebras of $L$, meaning subalgebras that are closed under arbitrary joins and meets from $L$, and $C\text{BSub} L$ be the set of complete Boolean subalgebras of $L$. Again, these are considered as posets, partially ordered by set inclusion.

For a von Neumann algebra $\mathcal{M}$ we can use the associative, but not necessarily commutative, product on $\mathcal{M}$ to define a commutative, but not necessarily associative product $\circ$ on $\mathcal{M}$, called the Jordan product, by setting

$$a \circ b = \frac{1}{2}(ab + ba).$$

Suppose $\varphi$ is a map between von Neumann algebras that is linear, bijective, and preserves the involution (adjoint) $\ast$. We say $\varphi$ is a $\ast$-isomorphism if it satisfies $\varphi(ab) = \varphi(a)\varphi(b)$; a $\ast$-antiisomorphism if it satisfies $\varphi(ab) = \varphi(b)\varphi(a)$; and a Jordan isomorphism if it satisfies $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$. The following is well known [21, 27].

**Proposition 2.5.** Every Jordan isomorphism $\eta : \mathcal{M} \rightarrow \mathcal{N}$ between von Neumann algebras $\mathcal{M}, \mathcal{N}$ can be decomposed as the sum of a $\ast$-isomorphism and a $\ast$-antiisomorphism.

More concretely, there are central projections $\hat{P}_1, \hat{P}_2 \in \mathcal{M}$ and $\hat{Q}_1, \hat{Q}_2 \in \mathcal{N}$ such that $\mathcal{M}$ and $\mathcal{N}$ are unitarily equivalent to $\mathcal{M}\hat{P}_1 \oplus \mathcal{M}\hat{P}_2$ and $\mathcal{N}\hat{Q}_1 \oplus \mathcal{N}\hat{Q}_2$, respectively, and $\eta|_{\mathcal{M}\hat{P}_1} : \mathcal{M}\hat{P}_1 \rightarrow \mathcal{N}\hat{Q}_1$ is a $\ast$-isomorphism, while $\eta|_{\mathcal{M}\hat{P}_2} : \mathcal{M}\hat{P}_2 \rightarrow \mathcal{N}\hat{Q}_2$ is a $\ast$-antiisomorphism.

It follows from [3] that there is a von Neumann algebra that is not $\ast$-isomorphic to its opposite, hence these two von Neumann algebras are Jordan isomorphic, but not $\ast$-isomorphic. So there can be two different associative noncommutative products on a weakly closed set of operators, giving different von Neumann algebras, but the same Jordan structure. So the associative noncommutative product on a von Neumann algebra cannot be recovered from the lattice of its subalgebras as a von Neumann algebra and its opposite will have precisely the same subalgebras. However, we will see that in the absence of type $I_2$ summands (and excluding the case $\mathcal{M} = \mathbb{C} \oplus \mathbb{C}$), the Jordan structure can be recovered. The following result by...
Dye [11], see also [13, Theorem 8.1.1], will be of key importance. We note that the uniqueness in the version of this result given below follows from the spectral theorem.

**Theorem 2.6.** Suppose $\mathcal{M}, \mathcal{N}$ are von Neumann algebras without type $I_2$ summands. Then for any OML-isomorphism $\psi : \text{Proj} \mathcal{M} \to \text{Proj} \mathcal{N}$ there is a unique Jordan $\ast$-isomorphism $\Psi : \mathcal{M} \to \mathcal{N}$ with $\Psi(p) = \psi(p)$ for each projection $p$ of $\mathcal{M}$.

The reader should consult [1, 12, 17, 21, 27] for basics on von Neumann algebras, [2] for lattice theory, and [22] for OMLs.

3. Main result

**Lemma 3.1.** Let $\mathcal{M}$ be a von Neumann algebra. Then there is an order-isomorphism $\Psi : F\text{AbSub} \mathcal{M} \to F\text{Sub} (\text{Proj} \mathcal{M})$ defined by setting $\Psi S = S \cap \text{Proj} \mathcal{M}$.

**Proof.** It follows from [1, Theorem 2.104] that the projections of any abelian subalgebra of $\mathcal{M}$ form a Boolean subalgebra of $\text{Proj} \mathcal{M}$. So $\Psi$ is indeed a map from $F\text{AbSub} \mathcal{M}$ to $F\text{Sub} (\text{Proj} \mathcal{M})$. Clearly $\Psi$ is order-preserving. Suppose $\Psi S \subseteq \Psi T$. As $S$ is a von Neumann algebra $S = (\text{Proj} S)'$, and similarly for $T$. Therefore $S = (\Psi S)' \subseteq (\Psi T)' = T$, showing $\Psi$ is an order-embedding.

Suppose $B$ is a finite Boolean algebra of projections in $\mathcal{M}$ with atoms $p_1, \ldots, p_n$, and consider the map $\Lambda : \mathbb{C}^n \to \mathcal{M}$ defined by setting $\Lambda(\lambda_1, \ldots, \lambda_n) = \sum_1^n \lambda_i p_i$. One easily sees $\Lambda$ is a normal, unital $\ast$-isomorphism, so by [1, Lemma 2.100] its image $\mathcal{S}$ is a von Neumann subalgebra of $\mathcal{M}$. Clearly $\mathcal{S}$ is an abelian, has finitely many projections, and $\Psi \mathcal{S} = B$. So $\Psi$ is onto.

**Remark.** While not needed for our results, it is natural to consider several questions related to the above result. It is easy to see that as above there is an order-embedding $\Psi : \text{Sub} \mathcal{M} \to C\text{Sub} (\text{Proj} \mathcal{M})$ that preserves all meets. A simple example with $\mathcal{M}$ being the bounded operators on $\mathbb{C}^2$ shows this map need not preserve joins or be onto. A more difficult argument, using the notion of Bade subalgebras and results from [24], shows there is an order-isomorphism $\Psi : A\text{bSub} \mathcal{M} \to C\text{BSub} (\text{Proj} \mathcal{M})$. The result above follows from this more general one, but is not needed here.

**Lemma 3.2.** For OMLs $L, M$, each order-isomorphism $\mu : F\text{Sub} L \to F\text{Sub} M$ extends uniquely to an isomorphism $\bar{\mu} : B\text{Sub} L \to B\text{Sub} M$.

**Proof.** We define an ideal of $F\text{Sub} L$ to be a downset $I$ of $F\text{Sub} L$ where any two elements of $I$ have a join, and this join belongs to $I$. For any element
of $BSub L$, we have $x \uparrow \cap FBS_{\sub} L = \{ z \in FBS_{\sub} L : z \subseteq x \}$ is an ideal of $FBS_{\sub} L$ and the join of this ideal in $BSub L$ is equal to $x$. Further, each ideal of $FBS_{\sub} L$ is of this form as can be easily seen from the compactness of finitely generated subalgebras in a subalgebra lattice.

Define $\bar{\mu}$ by setting $\bar{\mu}(x) = \bigvee \mu(x \uparrow \cap FBS_{\sub} L)$. This join is well defined as the image under the isomorphism $\mu$ of an ideal is an ideal. Clearly $\bar{\mu}$ is order preserving. Suppose $\bar{\mu}(x) \supseteq \bar{\mu}(y)$. Then for each $z \in x \uparrow \cap FBS_{\sub} L$ we have $\mu(z) \leq \bigvee \mu(y \downarrow \cap FBS_{\sub} L)$. Compactness then yields $z \leq y$ for each such $z$, giving $x \leq y$. Thus $\bar{\mu}$ is an order-embedding. To see $\bar{\mu}$ is onto, note each element $w$ of $BSub M$ is the join of an ideal $J$ of $FBS_{\sub} M$. The preimage $\mu^{-1}(J)$ is an ideal of $FBS_{\sub} L$, so has a join in $BSub L$. Then $\bar{\mu}(x) = w$, showing $\bar{\mu}$ is onto.

Clearly $\bar{\mu}$ extends $\mu$. If $\tilde{\mu}$ is another isomorphism from $BSub L$ to $BSub M$ extending $\mu$, then $\tilde{\mu}$ preserves joins, so $\tilde{\mu}(x) = \bigvee \mu(x \uparrow \cap FBS_{\sub} L) = \bar{\mu}(x)$. □

We are ready to provide our main result.

**Theorem 3.3.** Suppose $M, N$ are von Neumann algebras not isomorphic to $\mathbb{C} \oplus \mathbb{C}$ and without type $I_2$ summands, and suppose that $f : Ab_{\sub} M \rightarrow Ab_{\sub} N$ is an order-isomorphism. Then there is a unique Jordan $\ast$-isomorphism $F : M \rightarrow N$ with $f(S)$ equal to the image $F[S]$ for each $S$.

**Proof.** Consider a series of mappings, starting with the given

$$Ab_{\sub} M \xrightarrow{f} Ab_{\sub} N.$$  

We then restrict this to $FAb_{\sub} M$. Note that the members of $FAb_{\sub} M$ are precisely those members of $Ab_{\sub} M$ that have only finitely many elements beneath them, and similarly for $FAb_{\sub} N$. Thus this restriction $g$ is also an order-isomorphism.

$$FAb_{\sub} M \xrightarrow{g} FAb_{\sub} N.$$  

Lemma 3.1 gives order-isomorphisms $\Psi_M : FAb_{\sub} M \rightarrow FBS_{\sub} (Proj M)$ and $\Psi_N : FAb_{\sub} N \rightarrow FBS_{\sub} (Proj N)$ given by $\Psi_M(S) = S \cap Proj M$ and $\Psi_N(T) = T \cap Proj N$. It follows there is a unique order-isomorphism $h$ as below with $h(S \cap Proj M) = g(S) \cap Proj N$ for each $S \in FAb_{\sub} M$.

$$FBS_{\sub} (Proj M) \xrightarrow{h} FBS_{\sub} (Proj N).$$  

Then by Lemma 3.2 this extends uniquely to an order-isomorphism

$$BSub (Proj M) \xrightarrow{j} BSub (Proj N).$$
The main result of [16] says that if $L, M$ are omls without any 4-element blocks (a block is a maximal Boolean subalgebra), then for any order-isomorphism $\alpha : BSub L \to BSub M$ there is a unique oml-isomorphism $\beta : L \to M$ with $\alpha(D) = \beta[D]$ for each Boolean subalgebra $D$ of $L$. As $\mathcal{M}, \mathcal{N}$ are neither isomorphic to $\mathbb{C} \oplus \mathbb{C}$ nor to $B(\mathbb{C} \oplus \mathbb{C})$ (the latter is a von Neumann algebra of type $I_2$), there are no 4-element blocks in $Proj \mathcal{M}$ or $Proj \mathcal{N}$. So the map $j$ defined above gives a unique map $k$ as shown below with $j(D) = k[D]$ for each Boolean subalgebra $D$ of $Proj \mathcal{M}$.

$$Proj \mathcal{M} \xrightarrow{k} Proj \mathcal{N}.$$  

Finally, Theorem 2.6 gives a unique Jordan $\ast$-isomorphism $F$ as below extending $k$.

$$\mathcal{M} \xrightarrow{F} \mathcal{N}.$$  

**Claim 1.** If $S \in FAbsSub \mathcal{M}$ then $f(S) \cap Proj \mathcal{N} = F[S] \cap Proj \mathcal{N}$.

**Proof of Claim 1.** To see this, note that for such $S$,

$$f(S) \cap Proj \mathcal{N} = g(S) \cap Proj \mathcal{N} = h(S \cap Proj \mathcal{M}) = j(S \cap Proj \mathcal{M}) = k[S \cap Proj \mathcal{M}] = F[S \cap Proj \mathcal{M}] = F[S] \cap Proj \mathcal{N}$$

The first equality follows as $g$ is the restriction of $f$; the second by the definition of $h$; the third as $j$ extends $h$; the fourth by the definition of $k$; the fifth as $F$ extends $k$; and the sixth as $F$ restricts to a bijection between $Proj \mathcal{M}$ and $Proj \mathcal{N}$.  

**Claim 2.** If $S \in AbsSub \mathcal{M}$, then $F[S] \in AbSub \mathcal{N}$.

**Proof of Claim 2.** As $F$ is Jordan and $S$ is abelian, by [27, pg. 187] the restriction $F|S$ preserves the associative product. By [1, pg. 189] $F$ is a unital order-isomorphism, so it preserves monotone joins, and as $S$ is a von Neumann subalgebra of $\mathcal{M}$, the identical embedding of $S$ into $\mathcal{M}$ preserves monotone joins. So the composite $F|S$ preserves monotone joins, hence is a normal unital one-one $\ast$-homomorphism of $S$ into $\mathcal{N}$. So by [1, Lemma 2.100] the image $F[S]$ is a von Neumann subalgebra of $\mathcal{N}$ that is clearly abelian.  

**Claim 3.** If $S \in AbsSub \mathcal{M}$, then $f(S) = F[S]$.  


Proof of Claim 3. A projection $p$ belongs to $F[S]$ if, and only if, it belongs to $F[U]$ for some $U \subseteq S$ with $U \in \text{FAbSub } M$. The proof is essentially that of Lemma 3.1. By Claim 1, this is equivalent to $p$ belonging to $f(U)$ for some $U \subseteq S$ with $U \in \text{FAbSub } M$. As the members of $\text{FAbSub } M$ are exactly the members of $\text{AbSub } M$ with finitely many elements beneath them, it follows from $f$ being an order-isomorphism that $T = F[U]$ for some $U \subseteq S$ with $U \in \text{FAbSub } M$ if, and only if, $T \subseteq f(S)$ and $T \in \text{FAbSub } N$. So $p$ belonging to $F[S]$ is equivalent to $p$ belonging to $T$ for some $T \subseteq f(S)$ with $T \in \text{FAbSub } N$, so equivalent to $p$ belonging to $f(S)$. By Claim 2, $f(S)$ and $F[S]$ are von Neumann subalgebras of $N$, and they contain the same projections, so $f(S) = F[S]$. 

To conclude the proof of the theorem, it remains to show uniqueness. Suppose $G : M \to N$ is a Jordan $\ast$-isomorphism with $f(S) = G[S]$ for each $S \in \text{AbSub } M$. Using the spectral theorem, it follows that two Jordan $\ast$-isomorphisms from $M$ to $N$ agreeing on the projections must be equal. So it is enough to show that $F$ and $G$ agree on $\text{Proj } M$. From the uniqueness of the result in [16] it is enough to show $F[D] = G[D]$ for each Boolean subalgebra $D$ of $\text{Proj } M$, and by the uniqueness in Lemma 3.2 it is enough to show this for finite Boolean subalgebras $D$ of $\text{Proj } M$. Using Lemma 3.1, it is then enough to show $F[S \cap \text{Proj } M] = G[S \cap \text{Proj } M]$ for each $S \in \text{FAbSub } M$, and this is a direct consequence of the assumption that $F[S] = G[S]$. This shows $F = G$, and concludes the proof of the theorem.

We finally observe that the converse of the above result also holds (in fact, for arbitrary von Neumann algebras):

**Proposition 3.4.** Let $M, N$ be von Neumann algebras, and let $F : M \to N$ be a Jordan $\ast$-isomorphism. Then $F$ induces a unique order isomorphism $f : \text{AbSub } M \to \text{AbSub } N$ with $f(S)$ equal to the image $F[S]$ for each $S$.

**Proof.** It is well-known that a Jordan $\ast$-homomorphism $F : M \to N$ between von Neumann algebras preserves commutativity (see e.g. [27, 18]), so $F$ maps abelian subalgebras of $M$ to abelian subalgebras of $N$ in a bijective and order-preserving way. Hence, we obtain an order-isomorphism $f : \text{AbSub } M \to \text{AbSub } N$.  

4. Conclusions

There remain several directions for further research. First, it would be of interest to see if the Jordan structure of a $C^*$-algebra is determined by its poset of abelian $C^*$-subalgebras. In this direction we remark that it is known that the
lattice of $C^*$-subalgebras of an abelian $C^*$-algebra determines the $C^*$-algebra [23, Theorem 11]. Perhaps [25] may also be related to this question. [28] is concerned with abelian subalgebras of partial $C^*$-algebras and von Neumann algebras.

For a different direction, one might consider the matter of adding additional information to the poset $\text{AbSub} \ M$ in hopes of recovering the full von Neumann structure of $\mathcal{M}$, rather than just its Jordan structure. This seems very closely related to the subject of orientation theory, very nicely described in [1]. From the perspective of the topos approach, the natural question becomes whether orientations can be encoded by presheaves (contravariant, Set-valued functors) over $\text{AbSub} \ M$, or maybe by covariant functors.

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