Inequalities in De Morgan Systems I

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***Abstract*— In a Boolean algebra, every element has two canonical forms, its disjunctive normal form (BDNF) and its conjunctive normal form (BCNF). If these elements are viewed in the algebra consisting of the unit interval with operations min, max, and the usual negation, the inequality BDNF $\leq$ BCNF always holds. If the operations min and max in this algebra are replaced by a t-norm and its t-conorm dual to the usual negation, the resulting inequality holds sometimes and fails sometimes. This paper examines this phenomenon, expecially in the two variable case, which has been of interest to Turksen.**

***Index Terms*— Inequality, t-norm, normal form**

I. Introduction

In a Boolean algebra, every element has two canonical forms, its disjunctive normal form (BDNF) and its conjunctive normal form (BCNF). If these elements are viewed in the algebra consisting of the unit interval with operations min, max, and the usual negation, the inequality BDNF $\leq$ BCNF always holds. This fact is readily seen from the development of Kleene disjunctive and Kleene conjunctive normal forms [3], and holds for any finite number of variables. If the operations min and max in this algebra are replaced by a t-norm and its t-conorm dual with respect to the usual negation, the resulting inequality holds sometimes and fails sometimes. This paper examines this phenomenon, especially in the two variable case, the case which has been of interest to Turksen [5], [6], [7].

Boolean Disj. = Boolean Conj.	Short form
$(x\wedge y)\vee(x'\wedge y)\vee(x\wedge y')\vee(x'\wedge y')=1$	1
$0=(x\vee y)\wedge(x'\vee y)\wedge(x\vee y')\wedge(x'\vee y')$	0
$(x\wedge y)\vee(x\wedge y')\vee(x'\wedge y)=x\vee y$	$x\vee y$
$x'\wedge y'=(x\vee y')\wedge(x\vee y)\wedge(x'\vee y')$	$x'\wedge y'$
$(x'\wedge y)\vee(x\wedge y')\vee(x'\wedge y')=x'\vee y'$	$x'\vee y'$
$x\wedge y=(x\vee y)\wedge(x\vee y')\wedge(x'\vee y)$	$x\wedge y$
$(x\wedge y)\vee(x'\wedge y')\vee(x'\wedge y)=x'\vee y$	$x'\vee y$
$x\wedge y'=(x\vee y)\wedge(x\vee y')\wedge(x'\vee y')$	$x\wedge y'$
$(x\wedge y)\vee(x\wedge y')\vee(x'\wedge y')=x\vee y'$	$x\vee y'$
$x'\wedge y=(x\vee y)\wedge(x'\vee y)\wedge(x'\vee y')$	$x'\wedge y$
$(x\wedge y)\vee(x'\wedge y')=(x\vee y')\wedge(x'\vee y)$	$x\Leftrightarrow y$
$(x\wedge y')\vee(x'\wedge y)=(x\vee y)\wedge(x'\vee y')$	$x\Leftrightarrow y'$
$(x\wedge y)\vee(x\wedge y')=(x\vee y)\wedge(x\vee y')$	x
$(x'\wedge y)\vee(x'\wedge y')=(x\vee y)\wedge(x\vee y')$	x'
$(x\wedge y)\vee(x'\wedge y)=(x\vee y)\wedge(x'\vee y)$	y
$(x\wedge y')\vee(x'\wedge y')=(x\vee y')\wedge(x'\vee y')$	y'

The free Boolean algebra on two variables has exactly sixteen elements. Thus there are exactly sixteen disjunctive and sixteen conjunctive "normal forms" for a Boolean expression in two variables. These sixteen disjunctive and conjunctive normal forms in the variables x and y are shown above, where x' indicates "not x."

I. B. Turksen [7] raises the question of inequalities that may occur when $x,y\in[0,1]$, $\wedge$ is replaced by an arbitrary t-norm $\circ$, $x'=1-x$ and $\vee$ is replaced by the t-conorm $\diamond$ dual to $\circ$. In other words, when is it true that the following inequalities hold?

Boolean Disj. $\leq$ Boolean Conj.		
$(x\circ y)\diamond(x'\circ y)\diamond(x\circ y')\diamond(x'\circ y')\leq 1$	1	
$0\leq(x\diamond y)\circ(x'\diamond y)\circ(x\diamond y')\circ(x'\diamond y')$	0	
$(x\circ y)\diamond(x\circ y')\diamond(x'\circ y)\leq x\diamond y$	$x\diamond y$	
$x'\circ y'\leq(x\diamond y')\circ(x\diamond y)\circ(x'\diamond y')$	$x'\circ y'$	
$(x'\circ y)\diamond(x\circ y')\diamond(x'\circ y')\leq x'\diamond y'$	$x'\diamond y'$	
$x\circ y\leq(x\diamond y)\circ(x\diamond y')\circ(x'\diamond y)$	$x\circ y$	
$(x\circ y)\diamond(x'\circ y')\diamond(x'\circ y)\leq x'\diamond y$	$x'\diamond y$	
$x\circ y'\leq(x\diamond y)\circ(x\diamond y')\circ(x'\diamond y')$	$x\circ y'$	(1)
$(x\circ y)\diamond(x\circ y')\diamond(x'\circ y')\leq x\diamond y'$	$x\diamond y'$	
$x'\circ y\leq(x\diamond y)\circ(x'\diamond y)\circ(x'\diamond y')$	$x'\circ y$	
$(x\circ y)\diamond(x'\circ y')\leq(x\diamond y')\circ(x'\diamond y)$	$x\Leftrightarrow y$	
$(x\circ y')\diamond(x'\circ y)\leq(x\diamond y)\circ(x'\diamond y')$	$x\Leftrightarrow y'$	
$(x\circ y)\diamond(x\circ y')\leq(x\diamond y)\circ(x\diamond y')$	x	
$(x'\circ y)\diamond(x'\circ y')\leq(x\diamond y)\circ(x\diamond y')$	x'	
$(x\circ y)\diamond(x'\circ y)\leq(x\diamond y)\circ(x'\diamond y)$	y	
$(x\circ y')\diamond(x'\circ y')\leq(x\diamond y')\circ(x'\diamond y')$	y'	

To show that these inequalities do or do not hold for the sixteen BDNF $\leq$ BCNF inequalities, it suffices to consider the three inequalities

$x\circ y\leq(x\diamond y)\circ(x\diamond y')\circ(x'\diamond y)$	$x\circ y$	I	
$(x\circ y)\diamond(x\circ y')\leq(x\diamond y)\circ(x\diamond y')$	x	II	(2)
$(x\circ y)\diamond(x'\circ y')\leq(x\diamond y')\circ(x'\diamond y)$	$x\Leftrightarrow y$	III	

This is because each inequality must hold *for all* $x,y\in[0,1]$. Suppose the inequality labeled $x\circ y$ in (1) holds. Then clearly it holds for those labeled $x\circ y'$, $x'\circ y$, and $x'\circ y'$. Now note that taking the negation of both sides of the inequality for $x'\circ y'$ gives the inequality for $(x'\circ y')'=x\diamond y$ and hence for all of $x\diamond y'$, $x'\diamond y$ and $x'\diamond y'$. Clearly, starting with the inequality for any one of these eight yields the inequality for all the other seven. So any of these eight are equivalent to the inequality labeled I in (2). Each of the inequalities labeled x, y, x', y' in (1) implies each of the others and in particular are equivalent to the inequality labeled II in (2). Finally, the inequalities labeled $x\Leftrightarrow y$ and $x\Leftrightarrow y'$ are equivalent, and we denote the first by III.

We can express the t-conorm in terms of the t-norm and negation, so that we have only two operations to consider.

This gives us the three inequalities in the form

$x \circ y \leq (x' \circ y')' \circ (x' \circ y)' \circ (x \circ y')'$	I	
$\left((x \circ y)' \circ (x \circ y')'\right)' \leq (x' \circ y')' \circ (x' \circ y)'$	II	(3)
$\left((x \circ y)' \circ (x' \circ y')'\right)' \leq (x' \circ y)' \circ (x \circ y')'$	III	

So the basic question is "For which t-norms $\circ$ do these three inequalities hold, where $x' = 1 - x$?"

II. De Morgan Systems

The appropriate setting for these considerations is that of a De Morgan system on the unit interval.

Definition 1: A **De Morgan system** on the unit interval is an algebra

$$([0,1], \wedge, \vee, \circ, ', 0, 1)$$

where $\wedge$, and $\vee$ denote $\min$ and $\max$, $\circ$ is a t-norm, and $'$ is a negation.

By negation, we mean strong negation—a one-to-one onto order-reversing mapping whose square is the identity. One could include in the operations the dual of the t-norm with respect to the negation, but since it is determined by the other operations, that is optional. The inequalities with which we are concerned hold when $\circ = \wedge$ and $x' = 1 - x$, as we have pointed out. Our concern is with strict t-norms and with nilpotent ones. There is some general theory that is applicable.

Theorem 2: [1] Any De Morgan system with strict t-norm is isomorphic to $([0,1], \wedge, \vee, \cdot, ', 0, 1)$, where $\cdot$ is multiplication, and $'$ is some negation.

Theorem 3: [2] Any De Morgan system with nilpotent t-norm is isomorphic to $([0,1], \wedge, \vee, \blacktriangle, ', 0, 1)$, where $\blacktriangle$ is the Łukasiewicz t-norm and $'$ is some negation.

Applying Theorems 2 and 3 gives the three inequalities in the following two forms:

Strict De Morgan system with product and $'$		
$xy \leq (x'y')' (x'y)' (xy')'$	I	
$\left((xy)' (xy')'\right)' \leq (x'y')' (x'y)'$	II	(4)
$\left((xy)' (x'y')'\right)' \leq (x'y)' x'$	III	

Nilpotent DeMorgan sys. with Łukasiewicz and $'$
I $(x + y - 1) \vee 0$ $\leq (((x'+y' - 1) \vee 0)' + ((x' + y - 1) \vee 0)'$ $+ ((x + y' - 1) \vee 0)' - 1) \vee 0$
II $\left(\left(((x + y - 1) \vee 0)' + ((x + y' - 1) \vee 0)' - 1\right) \vee 0\right)'$ $\leq \left(((x' + y' - 1) \vee 0)' + ((x' + y - 1) \vee 0)' - 1\right) \vee 0$
III $\left(\left(((x + y - 1) \vee 0)' + ((x' + y' - 1) \vee 0)' - 1\right) \vee 0\right)'$ $\leq \left(((x' + y - 1) \vee 0)' + ((x + y' - 1) \vee 0)' - 1\right) \vee 0$

(5)

For the strict case then, as far as these inequalities are concerned, one may as well consider only multiplication for the t-norm, and vary the negation. Doing this, the question becomes "For what negations do the three inequalities (4) hold?" This seems like an easier question, since a negation is a function of only one variable, while a t-norm is a function of two variables. We suspect that there is no easy condition to require of a negation so that these inequalities hold. Similar remarks apply to the nilpotent case.

III. Inequalities for De Morgan systems

We show that the sixteen inequalities that hold for the t-norm $\min$ and the negation $x' = 1 - x$ hold for multiplication and Łukasiewicz, both with $x' = 1 - x$.

Proposition 4: The inequalities in (1) hold when $\circ$ is multiplication and $x' = 1 - x$.

Proposition 5: The inequalities in (1) hold when $\circ$ is the Łukasiewicz t-norm and $x' = 1 - x$

These are straightforward computations. In each case, it suffices to verify the three inequalities shown in (4) and (5).

For multiplication and $x' = 1 - x$, the first inequality becomes

$$\begin{aligned} xy &\leq (x'y')' (x'y)' (xy')' \\ &= (1 - (1 - x)(1 - y))(1 - (1 - x)y)(1 - x(1 - y)) \end{aligned}$$

or

$$\begin{aligned} 0 &\leq (1 - y)(1 - x)\left(x^2 y - x^2 y^2 + x y^2 - 2xy + x + y\right) \\ &= (1 - y)(1 - x)\left[\left(x^2 y(1 - y) + x(1 - y)\right)\right. \\ &\quad \left. + y(1 - x) + xy^2\right] \end{aligned}$$

which clearly holds. The second inequality becomes

$$\left((xy)' (xy')'\right)' \leq (x'y')' (x'y)'$$

or

$$\begin{aligned} 0 &\leq (x'y')' (x'y)' - \left((xy)' (xy')'\right)' \\ &= (1 - (1 - x)(1 - y))(1 - (1 - x)y) \\ &\quad - (1 - (1 - xy)(1 - x(1 - y))) \\ &= y\left(2x^2 + 1 - 2x\right)(1 - y) \\ &= y\left((x - 1)^2 + x^2\right)(1 - y) \end{aligned}$$

which clearly holds. The third inequality is

$$\left((x'y)' x'\right)' \leq (xy)' (x'y')'$$

or

$$\begin{aligned} 0 &\leq (xy)' (x'y')' - \left((x'y)' x'\right)' \\ &= (1 - xy)(1 - (1 - x)(1 - y)) \\ &\quad -1 + (1 - (1 - x)y)(1 - x(1 - y)) \\ &= 2xy - 2xy^2 - 2x^2 y + 2x^2 y^2 \\ &= 2xy(1 - y)(1 - x) \end{aligned}$$

which clearly holds. Therefore all the inequalities hold for the product t-norm with the negation $x' = 1 - x$.

For Łukasiewicz, the first inequality is

$$\begin{aligned} (x + y - 1) \vee 0 \;\leq\; & \{((x' + y' - 1) \vee 0)' + ((x' + y - 1) \vee 0)' \\ & + ((x + y' - 1) \vee 0)' - 2\} \vee 0 \end{aligned}$$

or

$$\begin{aligned} 0 &\leq \{(((x'+y'-1)\vee 0)' + ((x'+y-1)\vee 0)' \\ &\quad + ((x+y'-1)\vee 0)' - 2)\vee 0\} \\ &\quad - ((x+y-1)\vee 0) \\ &= \max\{0, 1-\max(0,1-x-y) - \max(0,-x+y) \\ &\quad - \max(0,x-y)\} - \max(0,x+y-1) \end{aligned}$$

Since the expression is symmetric in x and y, there are two cases to consider:

$$x+y \leq 1, \quad x \leq y$$

and

$$x+y \geq 1, \quad x \leq y.$$

For the first case, we want

$$\begin{aligned} 0 &\leq \max(0, 1-\max(0,1-x-y) - \max(0,-x+y) \\ &\quad - \max(0,x-y)) - \max(0,x+y-1) \\ &= \max(0, 1-(1-x-y)-(y-x)-0) - 0 \\ &= 2x \end{aligned}$$

which holds. For the second case, we want

$$\begin{aligned} 0 &\leq \max(0, 1-\max(0,1-x-y) - \max(0,-x+y) \\ &\quad - \max(0,x-y)) - \max(0,x+y-1) \\ &= \max(0, 1-(y-x)-0) - (x+y-1) \\ &= (1-y+x) - (x+y-1) \\ &= 2-2y \end{aligned}$$

which holds.

The second and third inequalities are similar, if tedious, and we omit the computations.

Additional examples may be found in [4], including examples where these inequalities fail.

If the inequalities are satisfied in a De Morgan system, then they are satisfied in any isomorphic De Morgan system. In particular, these inequalities are satisfied in any De Morgan system isomorphic to $([0,1], \wedge, \vee, \cdot, ', 0, 1)$, where as above $\cdot$ is multiplication and $x' = 1-x$.

For notational purposes, let $\alpha(x) = 1-x$. Let A be the group of all automorphisms of the unit interval with its usual order. That is, A consists of all one-to-one mappings of the unit interval onto itself that preserve order. Then every negation is of the form

$$\alpha_g = g^{-1}\alpha g$$

for some $g \in A$. Let C be the subgroup of A consisting of all elements z that commute with α, that is, for which $z\alpha = \alpha z$. Let $\mathbb{R}^+$ be the set of positive real numbers. For $x \in [0,1]$, defining $r(x) = x^r$ identifies $\mathbb{R}^+$ with a subgroup of A. Let

$$Cg\mathbb{R}^+ = \{cgr : c \in C, r \in \mathbb{R}^+\}.$$

Then the strict De Morgan system $([0,1], \wedge, \vee, \cdot, \alpha_g, 0, 1)$ is isomorphic to $([0,1], \wedge, \vee, \cdot, \alpha_h, 0, 1)$ if and only if

$$Cg\mathbb{R}^+ = Ch\mathbb{R}^+.$$

In particular, it can be seen by taking g to be the identity element of C, that the inequalities hold for the De Morgan systems $([0,1], \wedge, \vee, \cdot, \alpha_r, 0, 1)$ for $r \in \mathbb{R}^+$, in other words for Yager negations.

The nilpotent De Morgan system $([0,1], \wedge, \vee, \blacktriangle, \alpha_g, 0, 1)$ is isomorphic to $([0,1], \wedge, \vee, \blacktriangle, \alpha_h, 0, 1)$ if and only if

$$Cg = Ch.$$

This is equivalent to saying that $\alpha_g = \alpha_h$. All of this can be found in [1] and [2].

Consider De Morgan systems with t-norm multiplication. It is easy to find two non-isomorphic such De Morgan systems which both satisfy the inequalities. Negations are of two kinds: those for which the corresponding De Morgan system satisfies the three inequalities and those for which it does not. There does not seem to be a readily apparent group theoretic condition distinguishing the two kinds.

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