Inequalities in De Morgan Systems II

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***Abstract*— In [2], I. B. Turksen looked at the Boolean-disjunctive normal form of a term in two variables and the Boolean-conjunctive normal form of that term. He concluded that "for some cases of certain families," when these forms are interpreted in a De Morgan system by replacing the meet by a t-norm and the join by the dual t-conorm, the disjunctive normal form is contained in the conjunctive normal form. We give examples where some of these inequalities fail to hold for a large class of t-norms, and also give examples where they do hold.**

***Index Terms*— Inequality, t-norm, Boolean normal forms**

I. Introduction

Write $BDNF(t)$ and $BCNF(t)$ for the Boolean disjunctive and conjunctive normal forms of an expression t. In a Boolean algebra, it is always true that

$$BDNF(t) = BCNF(t).$$

In a Kleene or De Morgan algebra, in particular, for fuzzy sets and interval-valued fuzzy sets, the inequality

$$BDNF(t) \leq BCNF(t)$$

is always true. In fact, in [1] we showed a more general statement, namely

$$\begin{aligned} & BDNF(t) \leq KDNF(t) \leq DDNF(t) \\ = \quad & DCNF(t) \leq KCNF(t) \leq BCNF(t) \end{aligned}$$

where $KDNF(t)$ and $KCNF(t)$ denote the Kleene disjunctive and conjunctive normal forms of an expression t, and $DDNF(t)$ and $DCNF(t)$ denote the De Morgan disjunctive and conjunctive normal forms of an expression t. Moreover, these inequalities hold for any finite number of variables.

In a **De Morgan system**, consisting of the unit interval with its usual order along with a t-norm, negation, and the dual t-conorm, similar inequalities can be considered, obtained by replacing $\wedge$ by the t-norm, negation by a different choice of negation, and $\wedge$ by the dual t-conorm. This is a much more complicated situation, and the answer is that all of the inequalities hold in some De Morgan systems, and some or most of them fail in others. In this paper, we consider only the Boolean disjunctive and conjunctive normal forms. We will present here several examples where inequalities involving expressions in two variables fail to hold.

For the two-variable case, as was pointed out in [3], only the three inequalities

$$x \circ y \leq (x' \circ y')' \circ (x' \circ y)' \circ (x \circ y')' \qquad \text{(I)}$$

$$\left((x \circ y)' \circ (x \circ y')'\right)' \leq (x' \circ y')' \circ (x' \circ y)' \qquad \text{(II)}$$

$$\left((x \circ y)' \circ (x' \circ y')'\right)' \leq (x' \circ y)' \circ (x \circ y')' \qquad \text{(III)}$$

corresponding to the expressions $x \circ y$, x, and $x \Leftrightarrow y$ in a Boolean algebra, respectively, need to be considered, as the truth or falsity of the others will follow from the truth or falsity of these three. The three inequalities are sometimes referred to as AND, X, IFF, respectively, because those are terms the expressions represent in the Boolean case. Here, we have used the De Morgan laws to omit the appearance of the t-conorm.

We will first look in some detail at the situation with Frank t-norms. Then we give a specific example with Yager t-norms, showing that the situation is similar for that family. Finally, we give an example illustrating that there are t-norms and negations for which all of the inequalities fail, other than the two tautologies.

II. Frank t-norms

The De Morgan system $([0,1], \wedge, \vee, \circ_{F_b}, \alpha, 0, 1)$ with the Frank t-norm

$$x \circ_{F_b} y = \log_b \left(1 + \frac{(b^x - 1)(b^y - 1)}{b - 1}\right), \ b > 0, \ b \neq 1$$

and the negation $\alpha(x) = 1 - x$ is isomorphic to the system $([0,1], \wedge, \vee, \cdot, \sigma_b, 0, 1)$ with multiplication and the Sugeno negation

$$\sigma_b(x) = \frac{1 - x}{1 + (b - 1)x}$$

To see this, let $F_b(x) = \frac{b^x - 1}{b - 1}$. Then $F_b^{-1}(x) = \log_b(1 + by - y)$ and

$$\begin{aligned} x \circ_{F_b} y &= F_b^{-1}(F_b(x) F_b(y)) \\ \sigma_b(x) &= F_b^{-1} \alpha F_b(x) \end{aligned}$$

Then the function

$$F_b^{-1} : ([0,1], \wedge, \vee, \circ_{F_b}, \alpha, 0, 1) \to ([0,1], \wedge, \vee, \cdot, \sigma_b, 0, 1)$$

is an isomorphism. Isomorphisms preserve inequalities, so proving the $BDNF/BCDF$ inequalities to be true or false for the algebra $([0,1], \wedge, \vee, \cdot, \sigma_b, 0, 1)$ is equivalent to proving the same results for the algebra $([0,1], \wedge, \vee, \circ_{F_b}, \alpha, 0, 1)$.

To simplify some of the computations, we replace b by $a = b - 1$, and take

$$\eta_a(x) = \frac{1 - x}{1 + ax}, \ -1 < a, \ a \neq 0.$$

so that $\eta_a = \sigma_{a+1}$. Note that for $b=1$, $a=0$ and $\eta_0 = \alpha$, and

$$x \circ_{F_1} y = \lim_{b\to 1} \log_b \left(1 + \frac{(b^x-1)(b^y-1)}{b-1}\right) = xy$$

so the case $a=0$ was proved in [3].

With the Sugeno negation $\eta_a(x)$, the three inequalities we need to examine become

$$0 \le \frac{(1-x)(1-y)\,p(x,y,a)}{(-1-ax+axy-ay)^2(axy+1)} \tag{I$'$}$$

$$0 \le \frac{y(1-y)\,q(x,y,a)}{(1+axy)(1+a(x+y-xy))(1+ay(1+x^2(1-y)))} \tag{II$'$}$$

$$0 \le \frac{xy(1-x)(1-y)(1+a)}{(1+axy)^2}\frac{r(x,y,a)}{s(x,y,a)} \tag{III$'$}$$

where the expressions

$$\begin{aligned} p(x,y,a) &= x^2y^2(x+y-xy+1)a^3 \\ &\quad + xy(xy+2y+2x)a^2 \\ &\quad + \left(x^2y^2 - xy^2 + y^2 - x^2y + x^2 + 3xy\right)a \\ &\quad + x^2y + xy^2 - 2xy + x + y - x^2y^2 \end{aligned}$$

$$\begin{aligned} q(x,y,a) &= x^2y\left(y - 2yx + 1 + yx^2\right)a^3 \\ &\quad + x\left(x^3 - 2xy + 2y + 2x + 2x^2y - 2x^2\right)a^2 \\ &\quad + (x^4y - x^2y^2 + y - 2x^3y + 2x^3y^2 - x^4y^2 \\ &\quad + 3yx^2 + 2x^3 - 2x^2 - 2yx + 2x)a \\ &\quad -2x + 2x^2 + 1 \end{aligned}$$

$$\begin{aligned} r(x,y,a) &= xy(1+x+y-xy)a^2 \\ &\quad + ((1-y)(1-x)xy + 2x + 2y)a + 2 \end{aligned}$$

$$\begin{aligned} s(x,y,a) &= xya^2 + \left(x^2y^2 - x^2y - xy^2 + xy + x + y\right)a \\ &\quad + 1 \end{aligned}$$

are all polynomials in a.

A. The AND inequality

We show that the AND inequality (I$'$) holds for multiplication with Sugeno negations $\eta_a(x) = \frac{1-x}{1+ax}$, $a \ge -1/2$, $a \ne 0$, or equivalently, for Frank t-norms with $b \ge \frac{1}{2}$, $b \ne 1$ and negation $\alpha(x) = 1-x$. We also show that this inequality fails for all $-1 < a < -\frac{1}{2}$, that is, $0 < b < \frac{1}{2}$.

The AND inequality for multiplication, with the Sugeno negation $\frac{1-x}{1+ax}$, becomes

$$0 \le \frac{(1-x)(1-y)\,p(x,y,a)}{(1+ax+ay-axy)^2(1+axy)}$$

where

$$\begin{aligned} p(x,y,a) &= x^2y^2(x+y-xy+1)a^3 + xy(xy+2x+2y)a^2 \\ &\quad + (x^2y^2 + (1-x)y^2 + x^2(1-y) + 3xy)a \\ &\quad + xy(x+y-xy) + x + y - 2xy. \end{aligned}$$

Since

$$0 \le \frac{(1-x)(1-y)}{(1+ax+ay-axy)^2(1+axy)}$$

we need only examine the expression $p(x,y,a)$ to see if or when it is positive. For $a>0$, this is clearly the case, since all of the coefficients of $p(x,y,a)$ are positive. That is not the dividing line, however, and we look at the expression $p(x,y,a)$ for $a = d - \frac{1}{2}$

$$\begin{aligned} & p\left(x,y,d-\tfrac{1}{2}\right) \\ =\; & x^2y^2(x+y-xy+1)d^3 \\ & + \frac{1}{2}xy((4-3xy)(x+y-xy)+3xy)d^2 \\ & + \frac{1}{4}(-3x^3y^3 + 12xy + 3x^2y^3 - 12x^2y \\ & + 3x^3y^2 + 3x^2y^2 - 12xy^2 + 4x^2 + 4y^2)d \\ & + 2x^2y + 2xy^2 - \frac{1}{2}y^2 - \frac{1}{8}x^3y^2 - \frac{11}{8}x^2y^2 \\ & - \frac{1}{2}x^2 + y - \frac{7}{2}xy - \frac{1}{8}x^2y^3 + \frac{1}{8}x^3y^3 + x \end{aligned}$$

and show that the inequality holds for all $d \ge 0$ and thus for all $a \ge -\frac{1}{2}$. Then, for each a with $-1 < a < -\frac{1}{2}$, we will show that there is a pair x, y for which $p(x,y,a) < 0$.

To show that the inequality holds for all $d \ge 0$, we look at the coefficients and see that it suffices to get the four expressions

$$x + y - xy + 1$$

$$(4-3xy)(x+y-xy) + 3xy$$

$$\begin{aligned} & -3x^3y^3 + 12xy + 3x^2y^3 - 12x^2y + 3x^3y^2 \\ & \quad + 3x^2y^2 - 12xy^2 + 4x^2 + 4y^2 \end{aligned}$$

$$\begin{aligned} & 16x^2y + 16xy^2 - 4y^2 - x^3y^2 - 11x^2y^2 \\ & \quad - 4x^2 + 8y - 28xy - x^2y^3 + x^3y^3 + 8x \end{aligned}$$

to be positive for each x and y in $[0,1]$.

The first two are obviously positive. The following equality establishes the third

$$\begin{aligned} -\; & 3x^3y^3 + 12xy + 3x^2y^3 - 12x^2y + 3x^3y^2 + 3x^2y^2 \\ & - 12xy^2 + 4x^2 + 4y^2 \\ =\; & 9xy(1-x)(1-y) + 3xy(1-xy)(1-x)(1-y) \\ & + 4x^2\left(1-y^2\right) + 2y^2\left(1-x^2\right) + 2y^2 \end{aligned}$$

and the following equality establishes the fourth.

$$\begin{aligned} & 16x^2y + 16xy^2 - 4y^2 - x^3y^2 - 11x^2y^2 \\ & - 4x^2 + 8y - 28xy - x^2y^3 + x^3y^3 + 8x \\ =\; & 4(1-x)(1-y)(x+y-2xy) + 4x(1-y)(1-y+xy) \\ & + 4y(1-x) + x^2y^2(1-x)(1-y) \end{aligned}$$

To show that for $a < -1/2$ (or $b < 1/2$) the inequality fails for some x and y, consider the following.

$$\begin{aligned}
& p(x,y,b-1) \\
= \; & y^4\left(2y-y^2+1\right)(-1+b)^3+y^2\left(y^2+4y\right)(-1+b)^2 \\
& +\left(y^4-2y^3+5y^2\right)(-1+b)+2y^3-2y^2+2y-y^4 \\
= \; & -2y^4b^2+4y^3b^2+2y+2by^4-10by^3+5by^2 \\
& +2y^5b^3-6y^5b^2+6y^5b-7y^2-2y^5-2y^4+y^6 \\
& +8y^3+y^4b^3-y^6b^3+3y^6b^2-3y^6b \\
= \; & -y^4b^3\left(-2y+y^2-1\right)+y^3b^2(y-2)\left(3y^2-2\right) \\
& -y^2b\left(3y^2-5\right)(y-1)^2+y(y+2)(y-1)^4
\end{aligned}$$

The limit of this expression as $y\to 1$ is $b^2(2b-1)$, which is negative whenever $b<1/2$. This means that for x and y close to 1, $p(y,y,-1+b)$ is negative. Thus the inequality *fails* for the parameter b of the t-norm between 0 and $1/2$. For a picture showing the extent of the failure, here is a plot of the difference for the special case $y=x$, $b=\frac{1}{4}$.and here we zoom in on the negative part: In this picture, you can see that the inequality fails a small amount for x near 1.

Example 1: To illustrate, we give a numerical example, using the Frank t-norm and t-conorm with parameter $1/4$, which are defined by

$$\begin{aligned}
x\circ y &= \log_{\frac{1}{4}}\left(1+\frac{\left(\frac{1}{4^x}-1\right)\left(\frac{1}{4^y}-1\right)}{\frac{1}{4}-1}\right) \\
x\diamond y &= 1-\log_{\frac{1}{4}}\left[1+\frac{\left(\frac{1}{4^{1-x}}-1\right)\left(\frac{1}{4^{1-y}}-1\right)}{\frac{1}{4}-1}\right]
\end{aligned}$$

and the negation $x'=1-x$. To emphasize the setting of fuzzy sets, we look at expressions in A and B where $X=[0,1]$, $A(x)=x$ and $B(x)=\min\{1,9-9x\}$. Then, although $A\diamond B$, $A'\diamond B$, and $A\diamond B'$ all lie above $A\circ B$, their "Frank product" does not. To see this, we show that the inequality fails at $x=9/10$. We have

$$\begin{aligned}
A(9/10) &= B(9/10)=9/10 \\
A'(9/10) &= B'(9/10)=1/10
\end{aligned}$$

and

$$\begin{aligned}
& (A\circ B)(9/10) \\
= \; & \log_{\frac{1}{4}}\left(1+\frac{\left(\frac{1}{4^{9/10}}-1\right)\left(\frac{1}{4^{9/10}}-1\right)}{\frac{1}{4}-1}\right) \\
= \; & \log_{\frac{1}{4}}\left(-\frac{1}{3}-\frac{1}{12}2^{\frac{2}{5}}+\frac{2}{3}\sqrt[5]{2}\right) \\
\simeq \; & 0.816\,30
\end{aligned}$$

However,

$$\begin{aligned}
& ((A\diamond B)\circ(A'\diamond B)\circ(A\diamond B'))(9/10) \\
= \; & (9/10\diamond 9/10)\circ(1/10\diamond 9/10)\circ(9/10\diamond 1/10) \\
\simeq \; & 0.813\,64
\end{aligned}$$

Thus the inequality

$$\begin{aligned}
A\circ B &\leq (A'\circ B')'\circ(A'\circ B)'\circ(A\circ B')' \\
&= (A\diamond B)\circ(A'\diamond B)\circ(A\diamond B')
\end{aligned}$$

fails in this case, since $0.816\,30 \nleq 0.813\,64$.

B. The X inequality

We show that the X inequality (II) holds for multiplication with Sugeno negations $\eta(x)=\frac{1-x}{1+ax}$, $-1<a$, $a\neq 0$, or equivalently, for all Frank t-norms with $b>0$, $b\neq 1$, and negation $\alpha(x)=1-x$. This inequality for multiplication with the Sugeno negation $\frac{1-x}{1+ax}$ is

$$0\leq\frac{y(1-y)\,q(x,y,a)}{(1+axy)(1+a(x+y-xy))(1+ay(1+x^2(1-y)))}$$

where

$$\begin{aligned}
& q(x,y,a)=x^2y\left(1+y(1-x)^2\right)a^3 \\
& +x\left(2y+2x-2yx+2yx^2+x^3-2x^2\right)a^2 \\
& +(x^4y-x^2y^2+y-2x^3y+2x^3y^2 \\
& -x^4y^2+3yx^2+2x^3-2x^2-2yx+2x)a \\
& -2x+2x^2+1.
\end{aligned}$$

Since

$$0\leq\frac{y(1-y)}{(1+axy)(1+a(x+y-xy))(1+ay(1+x^2(1-y)))}$$

we need only examine the expression $q(x,y,a)$ to see if it is positive. The coefficients of a^3 and a^2 and the constant term are obviously positive. The coefficient of a is

$$\begin{aligned}
& x^4y-x^2y^2+y-2x^3y+2x^3y^2-x^4y^2 \\
& +3yx^2+2x^3-2x^2-2yx+2x \\
= \; & x^2(1-y)\left(x^2y+2x+y\right) \\
& +2x(1-x)(1-y)+y+2x^3y^2
\end{aligned}$$

which is clearly positive. So whenever the parameter $a\geq 0$, the expression is nonnegative.

To check for a between -1 and 0, we consider $q(1-x,1-y,-a)$ for a,x,y between 0 and 1 and show this is nonnegative. Written as a polynomial in y,

$$\begin{aligned}
& q(1-x,1-y,-a)=ax^2\left(1-a^2\right)(1-x)^2y^2 \\
& +a(-4a^2x^3+2ax^3+2x^3-x^4+2a^2x^4-2a^2x \\
& -2x+4ax+3a^2x^2-4ax^2+x^2+a^2+1-2a)y \\
& +(1-a)(a^2x^4-2a^2x^3-2a^2x+2a^2x^2+a^2 \\
& +2ax^3+4ax-2a-4ax^2-2x+2x^2+1)
\end{aligned}$$

The coefficient of y^2 is clearly positive. For the coefficient of y, we need to show that

$$\begin{aligned}
g(x,a) &= -4a^2x^3+2ax^3+2x^3-x^4+2a^2x^4-2a^2x \\
&\quad -2x+4ax+3a^2x^2-4ax^2+x^2+a^2+1-2a
\end{aligned}$$

is nonnegative for all $x,a\in(0,1)$, or equivalently, that $g(x,1-a)\geq 0$ for all $x,a\in(0,1)$. Now

$$\begin{aligned}
g(x,1-a) &= \left(2x^2+1\right)(x-1)^2a^2 \\
&\quad +x^2\left(4ax^2+x^2-6ax+2a\right)
\end{aligned}$$

and we need only consider

$$4ax^2+x^2-6ax+2a=2(1-2x)(1-x)a+x^2$$

For $x \leq \frac{1}{2}$, both terms are clearly nonnegative. For $x > \frac{1}{2}$ and $0 < a < 1$, we have

$$2(1-2x)(1-x)a \geq 2(1-2x)(1-x)$$

whence

$$\begin{aligned} 2(1-2x)(1-x)a + x^2 &\geq 2(1-2x)(1-x) + x^2 \\ &= 5\left(x - \frac{3}{5}\right)^2 + \frac{1}{5} \end{aligned}$$

which is nonnegative.

It remains to show that the "constant" term in $q(1-x, 1-y, -a)$ is nonnegative, or equivalently, that

$$\begin{aligned} &a^2x^4 - 2a^2x^3 - 2a^2x + 2a^2x^2 + a^2 \\ &+2ax^3 + 4ax - 2a - 4ax^2 - 2x + 2x^2 + 1 \end{aligned}$$

is nonnegative. But this expression can be written as

$$a^2x^4 + 2a(1-a)x^3 + 2\left(\tfrac{1}{2} - x(1-x)\right)(1-a)^2$$

which is clearly nonnegative for all $x, a \in (0,1)$.

C. The IFF inequality

We show that the IFF inequality (III) holds for multiplication with all Sugeno negations $\eta(x) = \frac{1-x}{1+ax}$, $a > -1$, $a \neq 0$, or equivalently, for all Frank t-norms with $b > 0$, $b \neq 1$ and negation $\alpha(x) = 1 - x$. Consider the inequality

$$0 \leq \frac{xy(1-x)(1-y)(1+a)r(x,y)}{(1+axy)^2 s(x,y)}$$

where

$$\begin{aligned} r(x,y,a) &= xy(1+x+y-xy)a^2 \\ &\quad + ((1-y)(1-x)xy + 2x + 2y)a + 2 \end{aligned}$$

$$\begin{aligned} s(x,y,a) &= xya^2 + (x^2y^2 - x^2y - xy^2 \\ &\quad + xy + x + y)a + 1 \end{aligned}$$

Since

$$0 \leq \frac{xy(1-x)(1-y)(1+a)}{(1+axy)^2}$$

we need only examine the expressions $r(x,y,a)$ and $s(x,y,a)$ to see if they have the same sign. We will show they are both nonnegative for all relevant values of a. To get r nonnegative, consider $a = -1 + b$:

$$\begin{aligned} &r(x,y,-1+b) \\ =\ &2 - 2y - 2x + 2yb - x^2y^2b^2 + 3x^2y^2b + xy^2b^2 \\ &-3xy^2b + x^2yb^2 - 3x^2yb - xyb + xyb^2 + 2xb \\ &-2x^2y^2 + 2xy^2 + 2x^2y \\ =\ &(-yx(-x-y+xy-1))b^2 + (2x + 2y + 3x^2y^2 - xy \\ &-3xy^2 - 3x^2y)b + -2(-1+x)(y-1)(xy-1) \end{aligned}$$

The coefficient of b^2 and the constant term are obviously positive. We need

$$2x + 2y + 3x^2y^2 - xy - 3xy^2 - 3x^2y$$

to be positive. The equality

$$\begin{aligned} &2x + 2y - 4xy + 3yx(1-x-y+xy) \\ =\ &2x + 2y + 3x^2y^2 - xy - 3xy^2 - 3x^2y \end{aligned}$$

establishes it. To get s always positive, just look at

$$\begin{aligned} &s(x,y,-1+b) \\ =\ &-xyb + xyb^2 - x^2y^2 + x^2y^2b + x^2y - x^2yb \\ &+xy^2 - xy^2b - x + xb - y + yb + 1 \\ =\ &xyb^2 + (1-xy)(x+y-xy)b \\ &+(1-x)(1-y)(1-xy) \end{aligned}$$

III. Yager t-norm with parameter 7

In this example with a nilpotent t-norm, we look only at the inequality shown in equation (I). Consider the Yager t-norm and t-conorm with parameter 7, defined by

$$\begin{aligned} x \circ y &= \max\left\{1 - \left((1-x)^7 + (1-y)^7\right)^{\frac{1}{7}}, 0\right\} \\ x \diamond y &= \min\left\{\left(x^7 + y^7\right)^{\frac{1}{7}}, 1\right\} \end{aligned}$$

This pair is dual with respect to the negation $x' = 1 - x$.

We look at expressions in A and B where $X = [0,1]$, $A(x) = x$ and $B(x) = \min\{1, 3 - 3x\}$. Then, although $A \diamond B$, $A' \diamond B$, and $A \diamond B'$ all lie above $A \circ B$, their "Yager product" does not. That is, inequality (I) fails for some x. To see this, we show that the inequality fails at $x = 3/4$.

$$\begin{aligned} A(3/4) &= B(3/4) = 3/4 \\ A'(3/4) &= B'(3/4) = 1/4 \end{aligned}$$

and

$$\begin{aligned} &(A \circ B)(3/4) \\ =\ &A(3/4) \circ B(3/4) \\ =\ &\max\left\{1 - \left((1-3/4)^7 + (1-3/4)^7\right)^{\frac{1}{7}}, 0\right\} \\ =\ &\max\left\{1 - \frac{1}{4}\sqrt[7]{2}, 0\right\} \\ \simeq\ &0.723\,98 \end{aligned}$$

Also,

$$\begin{aligned} (A \diamond B)(3/4) &= A(3/4) \diamond B(3/4) \\ &= \tfrac{3}{4}\sqrt[7]{2} \\ &\simeq 0.828\,07 \\ (A' \diamond B)(3/4) &= A'(3/4) \diamond B(3/4) \\ &= \tfrac{1}{4}\sqrt[7]{5472}\,2^{\frac{2}{7}} \\ &\simeq 0.750\,05 \\ (A \diamond B')(3/4) &= A(3/4) \diamond B'(3/4) \\ &= \tfrac{1}{4}\sqrt[7]{5472}\,2^{\frac{2}{7}} \\ &\simeq 0.750\,05 \end{aligned}$$

but taking t-norms gets a value *smaller* than $(A\circ B)(3/4)$:

$$\begin{aligned} & ((A\diamond B)\circ(A'\diamond B)\circ(A\diamond B'))(3/4) \\ = & (3/4\diamond 3/4)\circ(1/4\diamond 3/4)\circ(3/4\diamond 1/4) \\ = & \frac{3}{4}\sqrt[7]{2}\circ\frac{1}{4}\sqrt[7]{5472}2^{\frac{2}{7}}\circ\frac{1}{4}\sqrt[7]{5472}2^{\frac{2}{7}} \\ \simeq & 0.722\,62 \end{aligned}$$

Thus the inequality

$$\begin{aligned} A\circ B & \leq (A'\circ B')'\circ(A'\circ B)'\circ(A\circ B')' \\ & = (A\diamond B)\circ(A'\diamond B)\circ(A\diamond B') \end{aligned}$$

fails in this case.

For a picture showing the extent of the failure, here is a plot of both $\left((A'\circ B')'\circ(A'\circ B)'\circ(A\circ B')'\right)(x)$ and $(A\circ B)(x)$.They look remarkably similar, but zooming in on the high spot reveals the difference. As you can see, the extent of failure is small but distinct, for these fuzzy sets with this t-norm and negation.

IV. Example: "Flat" Negation

To show how badly these inequalities may fail, we show that there is a negation η for which, together with multiplication, all fourteen (or equivalently, all three groups) of the inequalities fail. That is, there is one example that violates all of the 16 inequalities that are not tautologies.

Any strict t-norm is isomorphic to multiplication, and for this example, we can assume the t-norm is multiplication. Choose a negation η such that $\eta(1/2)=1/2$, and $\eta(1/4)$ is very close to $1/2$. Here is a picture of such an η. Now let $x=y=1/2$. Then in the first inequality, we have

$$xy\leq\eta(\eta(x)\eta(y))\eta(\eta(x)y)\eta(x\eta(y))$$

or

$$\begin{aligned} 1/4 & \leq \eta(1/2)1/2)\eta((1/2)1/2)\eta((1/2)1/2) \\ & = (\eta(1/4))^3 \end{aligned}$$

But $\eta(1/4)$ is very close to $1/2$, so its cube is not $\geq 1/4$. So this inequality is violated.

For the second inequality, we have

$$\eta(\eta(xy)\eta(x\eta(y)))\leq\eta(\eta(x)\eta(y))\eta(\eta(x)y)$$

or

$$\eta(\eta(1/4)\eta(1/4))\leq\eta(1/4)\eta(1/4)$$

The left side is approximately $\eta(1/4)$ and the right side is $(\eta(1/4))^2$, which cannot be.

For the third inequality, we have

$$\eta(\eta(\eta(x)y)\eta(x\eta(y)))\leq\eta(xy)\eta(\eta(x)\eta(y))$$

or

$$\eta(\eta(1/4)\eta(1/4))\leq\eta(1/4)\eta(1/4)$$

which is the same as the previous case. We have proved the following.

Proposition 2: There are strict De Morgan systems for which all of the DNF $\leq$ CNF inequalities, other than the tautologies, fail simultaneously.

Example 3: For a specific example, consider the negation with the piecewise definition

$$\beta(x)=\begin{cases} -\frac{15}{4}x+1 & \text{if } x<1/8 \\ -\frac{1}{12}x+\frac{13}{24} & \text{if } 1/8<x<1/2 \\ -12x+\frac{13}{2} & \text{if } 1/2\leq x\leq 17/32 \\ -\frac{4}{15}x+\frac{4}{15} & \text{if } x>17/32 \end{cases}$$

This is the negation used for the preceding plot, and it produces $\left(\beta\left(\frac{1}{4}\right)\right)^3=\left(\frac{25}{48}\right)^3=\frac{15\,625}{110\,592}<\frac{1}{4}$. The t-conorm in this case is the following piecewise-linear surface:

It is easy to modify this example so that it works for nilpotent t-norms.

Example 4: Use β together with the t-normwhere $f:[0,1]\to\left[\frac{1}{16},1\right]$ is the function defined by

$$f(x)=\begin{cases} \frac{1}{16}+\frac{1}{2}x & \text{if } x<\frac{1}{8} \\ x & \text{if otherwise} \end{cases}$$

Then the three inequalities all fail at $x=y=\frac{1}{2}$. Thus we have the following result.

Proposition 5: There are nilpotent De Morgan systems for which all of the DNF $\leq$ CNF inequalities, other than the tautologies, fail simultaneously.

V. Summary

The question of the validity of the inequalities $BDNF\leq BCNF$ for strict and nilpotent t-norms is rather complicated. The answer depends entirely on the parameter in the case of familiar families of t-norms, and the answer is simply "no" in many cases.

Plots of the standard families of t-norms indicate they may all have regions of failure for at least one of the groups of inequalities. The particular cases shown in this paper or in [2] or [3] are summarized in the following table.

t-norm\ineq.	I	II	III
Minimum	true	true	true
Product	true	true	true
Łukasiewicz	true	true	true
Frank	true $b\geq\frac{1}{2}$; false $0<b<\frac{1}{2}$	true	true
Yager	false $b=7$		
Examples 3,4	false	false	false

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