Powers of t-norms

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Abstract

In this paper we consider the rth powers of a continuous t-norm for positive real numbers r, generalizing the notion of the diagonal (2nd power). We identify the increasing functions from the unit interval to itself that are realized as the rth power of some continuous t-norm. The strict and nilpotent cases are described in detail.

Keywords: t-norm, diagonal, power of t-norm, isomorphism

1 Introduction

This paper was inspired by the paper [7] of Mesiar and Navara concerning diagonals of continuous triangular norms, and an example we presented in [3] that called upon powers of strict t-norms. We generalize some of the results of Mesiar and Navara, and make the point that, in general, use of automorphisms of the unit interval as generators of t-norms enables one to take advantage of the underlying algebra. The group of automorphisms of the unit interval provides a natural mathematical framework in which to sort out what is going on and to describe the results. Yes, we are on our soapbox again, supporting use of automorphisms of the unit interval as generators and more generally, recommending taking advantage of a natural algebraic framework.

A **t-norm** is a binary operation[1] $\circ$ on the lattice $\mathbb{I} = ([0,1], \wedge, \vee, 0, 1)$ that is commutative, associative, non-decreasing, and satisfies $x \circ 1 = x$ for all $x \in [0,1]$. All of the t-norms we consider will be **convex**: whenever $x \circ y \leq c \leq x_1 \circ y_1$, there is an r between x and x_1 and an s between y and y_1 such that $c = r \circ s$. A t-norm is convex if and only if it is continuous in the usual sense of continuity of real functions. However, we describe this situation in terms of convexity since this definition can be made entirely within the context of the algebra $(\mathbb{I}, \circ)$. A t-norm is **Archimedean** if it is convex and $x \circ x < x$ for all $x \in (0,1)$ and **strict** if it is convex and $0 < x \circ x < x$ for all $x \in (0,1)$. A convex Archimedean t-norm that is not strict is **nilpotent**.

A function $f : [0,1] \to [0,1]$ is an **automorphism** of $\mathbb{I}$ if f is one-to-one and onto and preserves order. The set $\text{Aut}(\mathbb{I})$ of all automorphisms of $\mathbb{I}$ with ordinary composition of functions is a group. Associating the positive real number r with the automorphism $r(x) = x^r$ identifies the multiplicative group $\mathbb{R}^+$ of positive reals with a subgroup of $\text{Aut}(\mathbb{I})$.

The following representation theorem has its roots in the work of Abel [1]. A principal reference is [6]. See also [2] and [4]. We have stated the theorem in the natural algebraic framework of isomorphisms.

Theorem 1 *A t-norm $\circ$ is strict if and only if it is isomorphic to multiplication on the unit interval—that is, given a strict t-norm $\circ$, there is an automorphism f of $\mathbb{I}$ satisfying*

$$f(x \circ y) = f(x) f(y)$$

[1] The notation $T(x, y)$ is often used in place of $x \circ y$.

for all $x, y \in [0,1]$. *Two automorphisms* f *and* g *of* $\mathbb{I}$ *give isomorphisms with multiplication for the same t-norm if and only if there is a positive real number* s *such that* $f(x) = (g(x))^s$ *for all* $x \in [0,1]$.

Such an automorphism f is called a (multiplicative) **generator** for $\circ$, since $\circ$ is obtained from multiplication by the formula

$$x \circ y = f^{-1}(f(x) f(y))$$

Nilpotent t-norms also have multiplicative generators analogous to those for strict t-norms, but they are isomorphisms from $[0,1]$ to a proper subinterval of $[0,1]$, rather than automorphisms of $\mathbb{I}$. In order to stay in the context of $\mathrm{Aut}(\mathbb{I})$, we use the following theorem, which follows directly from the general representation theorem for convex, Archimedean t-norms (see [4]).

Theorem 2 *A t-norm* $\circ$ *is nilpotent if and only if it is isomorphic to the Łukasiewicz t-norm* $x \bullet y = (x + y - 1) \vee 0$, *that is, there is an automorphism* f *of* $\mathbb{I}$ *satisfying*

$$f(x \circ y) = (f(x) + f(y) - 1) \vee 0$$

for all $x, y \in [0,1]$. *The automorphism* f *is unique.*

We call the automorphism f the **L-generator** for $\circ$, since $\circ$ is obtained from the Łukasiewicz t-norm by the formula

$$x \circ y = f^{-1}((f(x) + f(y) - 1) \vee 0)$$

To illustrate the use of universal algebra methods in the more general setting of t-norms, in [3] we presented an example showing that the conjunctive logic obtained by using the unit interval with a strict Archimedean t-norm as the truth value algebra, is the same as the logic obtained by using the unit interval with a nilpotent Archimedean t-norm. This is equivalent to showing that the two algebras involved generate the same variety, where the variety generated by an algebra $\mathbb{A}$ is the collection of all algebras of the same type that satisfy all of the equations satisfied by $\mathbb{A}$. By a famous theorem of Birkhoff, this is the same as the collection of all algebras obtained as homomorphic images of subalgebras of direct products of $\mathbb{A}$. The example is the following.

Example 3 *All algebras* $(\mathbb{I}, \circ)$ *with* $\mathbb{I} = ([0,1], \wedge, \vee, 0, 1)$ *and* $\circ$ *a strict (continuous, Archimedean) t-norm are isomorphic to* $\mathbb{A} = (\mathbb{I}, \cdot)$ *where* $\cdot$ *is multiplication on the unit interval (Theorem 1). And all algebras* $(\mathbb{I}, \circ)$ *with* $\circ$ *a nilpotent (continuous, Archimedean) t-norm are isomorphic to* $\mathbb{B} = (\mathbb{I}, \bullet)$ *where* $x \bullet y = (x + y - 1) \vee 0$ *(Theorem 2). The two algebras* $\mathbb{A} = (\mathbb{I}, \cdot)$ *and* $\mathbb{B} = (\mathbb{I}, \bullet)$ *are not isomorphic. (The algebra* $\mathbb{B}$ *has a nonzero element* x *such that* $x \bullet x = 0$, *while* $\mathbb{A}$ *has no such element.) Remarkably, however, these two algebras generate the same variety. To show this, we show that* $\mathbb{B}$ *can be realized as a quotient of* $\mathbb{A}$ *and that* $\mathbb{A}$ *is isomorphic to a subalgebra of a product of copies of* $\mathbb{B}$.

Let $\mathcal{V}(\mathbb{A})$ *be the variety generated by* $\mathbb{A}$, $\mathcal{V}(\mathbb{B})$ *the variety generated by* $\mathbb{B}$, *and let* $a \in (0,1)$. *The relation* $\sim$ *on* $\mathbb{A}$ *given by* $x \sim y$ *if* $x, y \in [0,a]$ *is a congruence, and the quotient algebra* $\overline{\mathbb{A}} = \mathbb{A}/\sim$ *satisfies*

$$x \circ y = \begin{cases} x \cdot y & \text{if } x \cdot y > a \\ \bar{0} = \bar{a} & \text{if } x \cdot y \leq a \end{cases}$$

Using the bijection $f : [0,1] \to [a,1] : x \mapsto (1-a)x + a$, *it is easy to see that* $\circ$ *induces the t-norm*

$$x \diamond y = (a(x + y - 1) + (1-a)xy) \vee 0$$

on $\mathbb{I}$ and, in fact, $\overline{\mathbb{A}} \approx (\mathbb{I}, \diamond)$. There are numbers $x, y > a$ with $x \cdot y \leq a$, so this t-norm is nilpotent and $\overline{\mathbb{A}}$ is isomorphic to $\mathbb{B}$. Since varieties are closed under quotients, we see that $\mathcal{V}(\mathbb{B}) \subseteq \mathcal{V}(\mathbb{A})$.

Now we find a subalgebra of $\prod_{\mathbb{Z}^+} \mathbb{B}$ of the form $(\mathbb{I}, \circ)$ with $\circ$ a strict t-norm. For n, m positive integers, r a positive real number, and $y \in (0,1)$, the powers $y^{[n]}$, $y^{\left[\frac{1}{m}\right]}$, and $y^{[r]}$ are defined by

$$y^{[n]} = \overbrace{y \bullet y \bullet \cdots \bullet y}^{n \text{ times}} \qquad \left(y^{\left[\frac{1}{m}\right]}\right)^{[m]} = y \qquad y^{[r]} = \lim_{\frac{m}{n} \to r} \left(y^{\left[\frac{1}{n}\right]}\right)^{[m]}$$

Let $x_n \in \mathbb{B}$ with $x_n^{[n]} \neq 0$, $n \in \mathbb{Z}^+$. Set $a = (x_n) \in \prod_{\mathbb{Z}^+} \mathbb{B}$, $S = \left\{a^{[-\ln x]} : x \in [0,1]\right\}$. Then $\mathbb{S} = (S, \wedge, \vee, 0, 1, \circ)$, where $\wedge, \vee, 0, 1, \circ$ are the operations inherited from the coordinatewise operations on $\prod_{\mathbb{Z}^+} \mathbb{B}$, is a subalgebra of $\prod_{\mathbb{Z}^+} \mathbb{B}$ isomorphic to $\mathbb{A}$. Since varieties are closed under both products and subalgebras, we see that $\mathcal{V}(\mathbb{B}) = \mathcal{V}(\mathbb{A})$.

In this paper we make a more careful definition of powers and determine what functions occur as powers of strict or nilpotent t-norms.

2 Powers of strict t-norms

We first look at rth powers of a strict t-norm $\circ$ for arbitrary positive real numbers r. Given the representation for strict t-norms stated in Theorem 1, this notion is completely natural. That is, the algebra $([0,1], \wedge, \vee, \circ, 0, 1)$ is isomorphic to the algebra $([0,1], \wedge, \vee, \cdot, 0, 1)$ where $\cdot$ is ordinary multiplication. Thus rth powers in the ordinary sense carry over to the isomorphic system via the isomorphism and its inverse. We will fill in details in the following paragraphs. However, the deeper result that we will discuss is the converse: what functions are realized as the rth powers of some strict t-norm? We will also discuss questions regarding to what extent the t-norm is determined by one or more of its rth powers.

We observe informally in the following that a direct development of the concept of rth power of an arbitrary strict t-norm would be similar to developing the concept of real powers of real numbers. First, for positive integers n, the nth power of $\circ$ is defined informally by

$$x^{[n]} = \overbrace{x \circ x \circ \cdots \circ x}^{n \text{ times}}$$

The 2nd power of $\circ$ is $x^{[2]} = x \circ x$, commonly known as the **diagonal** of the t-norm $\circ$. Note that the diagonal of a strict t-norm is an automorphism of $\mathbb{I}$ that satisfies $x \circ x < x$ for $x \in (0,1)$.

For a strict t-norm, the notion of nth power naturally extends to nth roots by defining $x^{\left[\frac{1}{n}\right]}$ to be the unique solution to $\left(x^{\left[\frac{1}{n}\right]}\right)^{[n]} = x$ for positive integers n. This leads to the definition of positive rational powers by setting $x^{\left[\frac{m}{n}\right]} = \left(x^{\left[\frac{1}{n}\right]}\right)^{[m]}$ for positive integers m and n. This is independent of the representation of the rational $\frac{m}{n}$. To see this, consider $x^{\left[\frac{km}{kn}\right]}$. First,

$$\left(x^{\left[\frac{1}{kn}\right]}\right)^{[kn]} = x = \overbrace{x^{\left[\frac{1}{kn}\right]} \circ x^{\left[\frac{1}{kn}\right]} \circ \cdots \circ x^{\left[\frac{1}{kn}\right]}}^{kn \text{ times}}$$
$$= \overbrace{\left(x^{\left[\frac{1}{kn}\right]} \circ x^{\left[\frac{1}{kn}\right]} \circ \cdots \circ x^{\left[\frac{1}{kn}\right]}\right)}^{k \text{ times}} \circ \cdots \circ \overbrace{\left(x^{\left[\frac{1}{kn}\right]} \circ x^{\left[\frac{1}{kn}\right]} \circ \cdots \circ x^{\left[\frac{1}{kn}\right]}\right)}^{k \text{ times}}$$

where there are n groups of k factors. This demonstrates that $\left(x^{\left[\frac{1}{kn}\right]}\right)^{[k]} = x^{\left[\frac{1}{n}\right]}$. It is then easy to see that $\left(x^{\left[\frac{1}{kn}\right]}\right)^{[km]} = \left(\left(x^{\left[\frac{1}{kn}\right]}\right)^{[k]}\right)^{[m]} = \left(x^{\left[\frac{1}{n}\right]}\right)^{[m]}$.

In making a formal definition, we take advantage of the fact that all strict t-norms are isomorphic to multiplication where powers are already well known. Observe that for an isomorphism $f : (\mathbb{I}, \circ) \to (\mathbb{I}, \cdot)$, and for positive integers n,

$$f\left(\overbrace{x \circ x \circ \cdots \circ x}^{n \text{ times}}\right) = \overbrace{f(x) \cdot f(x) \cdot \cdots \cdot f(x)}^{n \text{ times}}$$

That is, $f\left(x^{[n]}\right) = (f(x))^n$, so that

$$x^{[n]} = f^{-1}\left((f(x))^n\right).$$

By arguments similar to the above,

$$x^{\left[\frac{m}{n}\right]} = f^{-1}\left((f(x))^{\frac{m}{n}}\right)$$

for all positive integers m, n. We extend this to arbitrary positive real numbers, calling on the continuity of the isomorphism.

Proposition 4 *Given a strict t-norm $\circ$, the function $x^{[r]} = f^{-1}rf(x)$ is independent of the choice of isomorphism $f : (\mathbb{I}, \circ) \to (\mathbb{I}, \cdot)$.*

Proof. By Theorem 1, any other isomorphism is of the form sf for some positive real number s. Thus

$$\begin{aligned}(sf)^{-1}\left(sf(x)^r\right) &= f^{-1}\left(s^{-1}\left(f(x)^{rs}\right)\right) \\ &= f^{-1}\left((f(x))^{rss^{-1}}\right) \\ &= f^{-1}\left((f(x))^r\right)\end{aligned}$$

■

Definition 5 *Given a strict t-norm $\circ$ and a positive real number r, the* r***th power of*** $\circ$ *is defined to be the function*

$$x^{[r]} = f^{-1}\left((f(x))^r\right) = f^{-1}rf(x)$$

where f is any isomorphism of $\circ$ with multiplication.

Thus we see that the theory already provides a mechanism for defining arbitrary rth powers for $r \in \mathbb{R}^+$. Note, in particular, that the rth power of a t-norm $\circ$ is an automorphism $\delta = f^{-1}rf$ of $\mathbb{I}$.

Powers of strict t-norms satisfy the usual laws of exponents.

Theorem 6 *Given a strict t-norm $\circ$, the powers of $\circ$ satisfy the following.*

1. $\left(x^{[r]}\right)^{[s]} = x^{[rs]}$ *for all $r, s \in \mathbb{R}^+$.*
2. $x^{[r]} \circ x^{[s]} = x^{[r+s]}$ *for all $r, s \in \mathbb{R}^+$.*

3. $x^{[r]} = \lim_{\frac{m}{n}\to r} \left(x^{\left[\frac{1}{n}\right]}\right)^{[m]}$ *where* $n, m \in \mathbb{Z}^+$.

Proof. Let f be an isomorphism between $\circ$ and multiplication. From Definition 5,

$$\begin{aligned}\left(x^{[r]}\right)^{[s]} &= f^{-1}\left(\left(f\left(x^{[r]}\right)\right)^s\right) = f^{-1}\left(\left(f\left(f^{-1}\left((f(x))^r\right)\right)\right)^s\right) \\ &= f^{-1}\left(\left((f(x))^r\right)^s\right) = f^{-1}\left((f(x))^{rs}\right) = x^{[rs]}.\end{aligned}$$

Similarly,

$$\begin{aligned}x^{[r]} \circ x^{[s]} &= f^{-1}\left(f\left(x^{[r]}\right) f\left(x^{[s]}\right)\right) = f^{-1}\left(f\left(f^{-1}\left((f(x))^r\right)\right) f\left(f^{-1}\left((f(x))^s\right)\right)\right) \\ &= f^{-1}\left((f(x))^r (f(x))^s\right) = f^{-1}\left((f(x))^{r+s}\right) = x^{[r+s]}.\end{aligned}$$

In particular

$$\left(x^{\left[\frac{1}{n}\right]}\right)^{[m]} = x$$

for all $m, n \in \mathbb{Z}^+$. Part (3) follows from the continuity of the functions f , r, and $\frac{m}{n}$ since

$$\begin{aligned}x^{[r]} &= f^{-1} r f(x) \\ \left(x^{\left[\frac{1}{n}\right]}\right)^{[m]} &= f^{-1} \frac{m}{n} f(x)\end{aligned}$$

so that

$$\lim_{\frac{m}{n}\to r} f^{-1} \frac{m}{n} f(x) = f^{-1} r f(x)$$

■

The question now arises—*which* automorphisms of $\mathbb{I}$ occur as powers of strict t-norms? Mesiar and Navara proved in [7] that any automorphism δ of $\mathbb{I}$ satisfying $\delta(x) < x$ for all $x \in (0,1)$ is the diagonal of a strict t-norm—that is, there exists $f \in \mathrm{Aut}(\mathbb{I})$ such that $\delta(x) = f^{-1}(f(x) f(x)) = x \circ_f x$. The following theorem generalizes this to arbitrary rth powers.

Theorem 7 *Let* $r \in \mathbb{R}^+$. *A function* $\delta : [0,1] \to [0,1]$ *is the* r*th power for some strict t-norm* $\circ$ *if and only if* $\delta \in \mathrm{Aut}(\mathbb{I})$ *and one of the following holds:*

1. $r > 1$ *and* $\delta(x) < x$ *for all* $x \in (0,1)$.
2. $r = 1$ *and* $\delta(x) = x$ *for all* $x \in (0,1)$.
3. $r < 1$ *and* $\delta(x) > x$ *for all* $x \in (0,1)$.

Proof. If δ is the rth power for a strict t-norm $\circ$, then $\delta = f^{-1} r f \in \mathrm{Aut}(\mathbb{I})$ where f is a generator for the t-norm. Let $x \in (0,1)$. If $r > 1$, then $\delta(x) = f^{-1}\left((f(x))^r\right) < f^{-1}(f(x)) = x$ since $f(x)^r < f(x)$. If $r = 1$ then $\delta(x) = f^{-1} f(x) = x$. And if $r < 1$, then $\delta(x) = f^{-1}\left((f(x))^r\right) > f^{-1}(f(x)) = x$ since $f(x)^r > f(x)$.

Now suppose $\delta \in \mathrm{Aut}(\mathbb{I})$, $r > 1$ and $\delta(x) < x$ for all $x \in (0,1)$. We want to find a function $f \in \mathrm{Aut}(\mathbb{I})$ such that $\delta = f^{-1} r f$. By Theorem 1, for any $s \in R^+$, automorphisms f and sf determine the same strict t-norm . Taking $s = \ln(1/2) / \ln f(1/2)$ gives an automorphism sf with fixed point $1/2$, so we may assume that $f\left(\frac{1}{2}\right) = \frac{1}{2}$. This implies that, $f\left(\delta\left(\frac{1}{2}\right)\right) = rf\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^r$. Since $\delta(x) < x$ for all $x \in (0,1)$, there is a sequence

$$\begin{aligned}0 &< \cdots < \delta^n\left(\tfrac{1}{2}\right) < \delta^{n-1}\left(\tfrac{1}{2}\right) < \cdots < \delta\left(\tfrac{1}{2}\right) < \tfrac{1}{2} \\ &< \delta^{-1}\left(\tfrac{1}{2}\right) < \cdots < \delta^{-n}\left(\tfrac{1}{2}\right) < \delta^{-(n+1)}\left(\tfrac{1}{2}\right) < \cdots < 1\end{aligned}$$

and since δ is continuous, this sequence converges to 0 on the left and 1 on the right. Since $f\delta = rf$, $f\delta^2 = rf\delta = r^2 f$, and by induction, $f\delta^n = r^n f$ for positive integers n. Also $f\delta^{-1} = r^{-1} f$ and by induction $f\delta^{-n} = r^{-n} f$, so for any integer n, we have $f\delta^n = r^n f$.

Let g be any isomorphism $\left[\delta\left(\frac{1}{2}\right), \frac{1}{2}\right] \approx \left[\left(\frac{1}{2}\right)^r, \frac{1}{2}\right]$. Now

$$[0,1] = \bigcup_{n\in\mathbb{Z}} \left[\delta^n\left(\frac{1}{2}\right), \delta^{n-1}\left(\frac{1}{2}\right)\right]$$

and for any $x \in (0,1)$ there is a (unique) integer n such that $\delta\left(\frac{1}{2}\right) < \delta^n(x) \leq \frac{1}{2}$. Define $f(x) = \left(g\left(\delta^n(x)\right)\right)^{r^{-n}}$ (where n depends on x). We need to show that $f \in \mathrm{Aut}(\mathbb{I})$ and $\delta = f^{-1} r f$.

First we show that $f\delta = rf$. For n such that $\delta\left(\frac{1}{2}\right) < \delta^n(x) = \delta^{n-1}(\delta(x)) \leq \frac{1}{2}$, we have $f\delta(x) = \left(g\left(\delta^{n-1}(\delta(x))\right)\right)^{r^{-(n-1)}}$ while $f(x) = \left(g\left(\delta^n(x)\right)\right)^{r^{-n}}$, so $f\delta = rf$. Now on each interval $\left[\delta^n\left(\frac{1}{2}\right), \delta^{n-1}\left(\frac{1}{2}\right)\right]$ we have $f = r^{-n} g \delta^n$ is the composition of convex increasing functions, so f is convex and increasing on each of these intervals. At the endpoints of these intervals, we have

$$f\left(\delta^n\left(\tfrac{1}{2}\right)\right) = \left(g\left(\delta^{-n}\left(\delta^n\left(\tfrac{1}{2}\right)\right)\right)\right)^{r^n} = \left(g\left(\tfrac{1}{2}\right)\right)^{r^n}$$

and

$$\begin{aligned} f\left(\delta^n\left(\tfrac{1}{2}\right)\right) &= f\left(\delta\left(\delta^{n-1}\left(\tfrac{1}{2}\right)\right)\right) = rf\left(\delta^{n-1}\left(\tfrac{1}{2}\right)\right) \\ &= r\left(g\left(\delta^{-(n-1)}\left(\delta^{n-1}\left(\tfrac{1}{2}\right)\right)\right)\right)^{r^{n-1}} = r\left(g\left(\tfrac{1}{2}\right)\right)^{r^{n-1}} = \left(g\left(\tfrac{1}{2}\right)\right)^{r^n} \end{aligned}$$

so f is continuous at the points $\delta^n\left(\frac{1}{2}\right)$. It follows that $f \in \mathrm{Aut}(\mathbb{I})$. Since $f\delta = rf$, δ is the rth power of the strict t-norm defined by $x \circ y = f^{-1}(f(x) f(y))$.

Condition (2) is trivial, since the identity function is the 1st power for all t-norms. The proof for condition (3) is like the proof of (1), reversing appropriate inequalities. ■

We now consider the question of uniqueness of the strict t-norm, given the rth power. As a direct consequence of the proof of Theorem 1, if *all* the rth powers are known, the strict t-norm is uniquely determined. (For an outline of that proof see, for example, Theorem 9 in [2].) We will show, however, that there are many strict t-norms that have the same rth power for any given power $r \neq 1$.

The classification of these t-norms depends on special subgroups, called centralizers, of the automorphism group $\mathrm{Aut}(\mathbb{I})$ of the lattice $\mathbb{I} = ([0,1], \wedge, \vee, 0, 1)$.

Definition 8 *The* ***centralizer*** *of a positive real number r in $\mathrm{Aut}(\mathbb{I})$ is the subgroup*

$$Z(r) = \{\gamma \in \mathrm{Aut}(\mathbb{I}) : r\gamma = \gamma r\}$$

In other words, the centralizer of r consists of those automorphisms γ satisfying the identity $(\gamma(x))^r = \gamma(x^r)$ for all $x \in [0,1]$.

Theorem 9 *Let r be a positive real number. Two automorphisms $f, g \in \mathrm{Aut}(\mathbb{I})$ generate strict t-norms having the same rth power if and only if fg^{-1} is in the centralizer of r in $\mathrm{Aut}(\mathbb{I})$.*

Proof. The automorphisms f and g generate strict t-norms with the same rth power if and only if

$$f^{-1}\left(f(x)^r\right) = g^{-1}\left(g(x)^r\right)$$

for all $x \in [0,1]$. Replacing x by $g^{-1}(x)$ gives the equivalent condition

$$g\left(f^{-1}\left(f\left(g^{-1}(x)\right)^{r}\right)\right) = \left(g\left(g^{-1}(x)\right)\right)^{r} = x^{r}$$

which is equivalent to

$$\left(fg^{-1}(x)\right)^{r} = fg^{-1}(x^{r})$$

meaning fg^{-1} is in the centralizer of r in $\operatorname{Aut}(\mathbb{I})$. ■

This effectively reduces the problem of describing which strict t-norms have the same rth power to the problem of describing the centralizer of r in $\operatorname{Aut}(\mathbb{I})$. First note that $\mathbb{R}^{+} \subseteq Z(r)$ for any $r \in \mathbb{R}^{+}$, simply because multiplication is commutative. Also, if $\gamma \in Z(r)$, then $s\gamma \in Z(r)$ for any $s \in \mathbb{R}^{+}$ since

$$r(s\gamma) = (rs)\gamma = (sr)\gamma = s(r\gamma) = s(\gamma r) = (s\gamma)r$$

By Theorem 1, $f, g \in \operatorname{Aut}(\mathbb{I})$ generate the same strict t-norm if and only if $fg^{-1} \in \mathbb{R}^{+}$. We will show that $\mathbb{R}^{+} \subsetneqq Z(r)$, and that $Z(r)$ is, in fact, much larger than $\mathbb{R}^{+}$.

Let $u \in (0,1)$ and $h \in Z(r)$. Now $(h(u))^{s} = u$ if and only if $s = \ln u / \ln hf(u) \in \mathbb{R}^{+}$, so $\gamma = sh$ has the fixed point u. Now we have

$$h = s^{-1}\gamma$$

is the composition of an element γ of the centralizer of r with fixed point u and an element $s^{-1} \in \mathbb{R}^{+}$, that is

$$Z(r) = \left\{t\gamma : t \in \mathbb{R}^{+}, \gamma \in Z(r), \gamma(u) = u\right\} \qquad (*)$$

This reduces the problem further to that of describing the elements γ of the centralizer with fixed point u. We will find those $\gamma \in Z(r)$ such that $\gamma(u) = u$, for a fixed $u \in (0,1)$ by showing that the centralizer elements with the fixed point u are built up from "proportional" copies of an automorphism φ of the subinterval $[u^{r}, u]$, and every such φ gives a centralizer of r with fixed point u.

Theorem 10 *An automorphism γ of $\mathbb{I}$ with fixed point u is in the centralizer of $r \in \mathbb{R}^{+}$ if and only if one of the following holds.*

1. *$r > 1$ and there is an automorphism $\varphi \in \operatorname{Aut}([u^{r}, u], \leq)$ such that*

$$\gamma(x) = \left(\varphi\left(x^{r^{-n}}\right)\right)^{r^{n}}$$

for $x \in \left[u^{r^{n+1}}, u^{r^{n}}\right]$, n an integer.

2. *$r < 1$ and there is an automorphism $\varphi \in \operatorname{Aut}([u, u^{r}], \leq)$ such that*

$$\gamma(x) = \left(\varphi\left(x^{r^{-n}}\right)\right)^{r^{n}}$$

for $x \in \left[u^{r^{n}}, u^{r^{n+1}}\right]$, n an integer.

3. *$r = 1$*

Proof. Suppose $\gamma \in Z(r)$ and $\gamma(u) = u$. We can assume that $r \neq 1$. Since γ commutes with r,

$$\gamma(u^{r}) = (\gamma(u))^{r} = u^{r}$$

so γ fixes u^r. We now show that γ fixes any number of the form u^{r^n}, with n an integer. First, inducting on n for n positive,

$$\gamma\left(u^{r^{n+1}}\right)=\gamma\left(\left(u^{r^n}\right)^r\right)=\left(\gamma\left(u^{r^n}\right)\right)^r=\left(u^{r^n}\right)^r=u^{r^{n+1}}$$

For negative exponents,

$$\gamma\left(u\right)=\gamma\left(\left(u^{r^{-1}}\right)^r\right)=\left(\gamma\left(u^{r^{-1}}\right)\right)^r=u=\left(u^{r^{-1}}\right)^r$$

so that $\gamma\left(u^{r^{-1}}\right)=u^{r^{-1}}$ and by an induction similar to above, $\gamma\left(u^{r^{-n}}\right)=u^{r^{-n}}$ implies

$$\gamma\left(u^{r^{-(n+1)}}\right)=\gamma\left(\left(u^{r^{-n}}\right)^{r^{-1}}\right)=\left(\gamma\left(u^{r^{-n}}\right)\right)^{r^{-1}}=\left(u^{r^{-n}}\right)^{r^{-1}}=u^{r^{-(n+1)}}$$

Now we look at the behavior of γ at other points in the unit interval. In the case $r>1$, since γ is an automorphism of $\mathbb{I}$ and fixes the points u^r and u, the restriction of γ to the interval $[u^r,u]$ induces an automorphism φ of the interval $[u^r,u]$. For $x\in\left[u^{r^2},u^r\right]$, $x^{\frac{1}{r}}\in[u^r,u]$. Moreover, since $\gamma\in Z\left(r\right)$,

$$\gamma\left(x\right)=\gamma\left(\left(x^{\frac{1}{r}}\right)^r\right)=\left(\gamma\left(x^{\frac{1}{r}}\right)\right)^r.$$

So the values of γ on the interval $\left[u^{r^2},u^r\right]$ are completely determined by the values of γ on the interval $[u^r,u]$. Similarly, if $x\in\left[u^{r^{n+1}},u^{r^n}\right]$ then $\left(u^{r^{n+1}}\right)^{r^{-n}}=u^r\leq x^{r^{-n}}\leq\left(u^{r^n}\right)^{r^{-n}}=u$ and by induction

$$\gamma\left(x\right)=\left(\gamma\left(x^{r^{-n}}\right)\right)^{r^n}=\left(\varphi\left(x^{r^{-n}}\right)\right)^{r^n}$$

In the case $r<1$ the proof is obtained by reversing the endpoints in the above intervals.

For the converse, take $r>1$ and assume φ is any automorphism of the interval $[u^r,u]$. for any $x\in(0,1)$ there is a (unique) integer n such that $x\in\left(u^{r^{n+1}},u^{r^n}\right]$. Extend φ to a function γ on $[0,1]$ by the formula

$$\gamma\left(x\right)=\left(\varphi\left(x^{r^{-n}}\right)\right)^{r^n}$$

for $x\in\left(u^{r^{n+1}},u^{r^n}\right]$ and $\gamma\left(1\right)=1$, $\gamma\left(0\right)=0$. It is straightforward to show that γ is an automorphism of $[0,1]$ satisfying $\gamma\left(x^r\right)=\left(\gamma\left(x\right)\right)^r$ for all $x\in[0,1]$ so $\gamma\in Z\left(r\right)$. Also $\gamma\left(u\right)=u$ since $\varphi\left(u\right)=u$. In the case $0<r<1$, assume φ is any automorphism of the interval $[u,u^r]$ and a similar argument goes through for intervals $\left[u^{r^n},u^{r^{n+1}}\right]$, n an integer. The statement is trivial when $r=1$. This proves the theorem. ■

Corollary 11 *The set $Z_u\left(r\right)$ of automorphisms γ of $\mathbb{I}$ in the centralizer of $r\in\mathbb{R}^+$ with fixed point $u\in(0,1)$ is a subgroup of* $\operatorname{Aut}(\mathbb{I})$*. For $r>1$, it is isomorphic to* $\operatorname{Aut}\left([u^r,u],\leq\right)$ *via the restriction map $\gamma\mapsto\gamma\upharpoonright[u^r,u]$, and for $r\in(0,1)$ it is isomorphic to* $\operatorname{Aut}\left([u,u^r],\leq\right)$ *via the restriction map $\gamma\mapsto\gamma\upharpoonright[u,u^r]$, Moreover,*

$$Z\left(r\right)=\left\{s\gamma:\gamma\in Z_u\left(r\right),s\in\mathbb{R}^+\right\}$$

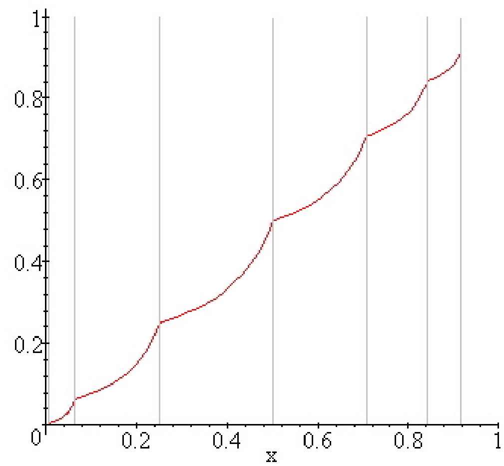
and $s\gamma=t\beta$ if and only if $\gamma=\beta$ and $s=t$.

Proof. We give the proof for $r > 1$. The proof for $r < 1$ is similar. If $\gamma, \beta \in Z_u(r)$, then by Theorem 10 the restrictions $\gamma \upharpoonright [u^r, u]$ and $\beta \upharpoonright [u^r, u]$ belong to $\mathrm{Aut}([u^r, u], \leq)$ and it is easy to see that $(\gamma \upharpoonright [u^r, u]) \circ (\beta \upharpoonright [u^r, u]) = (\gamma \circ \beta) \upharpoonright [u^r, u]$. Thus $Z_u(r) \to \mathrm{Aut}([0,1], \leq) : \gamma \mapsto \gamma \upharpoonright [u^r, u]$ is a homomorphism. Theorem 10 gives an inverse homomorphism as well, so these groups are isomorphic.

By Equation (*), every element of $Z(r)$ is of the form $s\gamma$, $\gamma \in Z_u(r), s \in \mathbb{R}^+$. Suppose $s\gamma = t\beta$. Then $\gamma\beta^{-1} = s^{-1}t \in Z_u(r) \cap \mathbb{R} = \{1\}$. So $\gamma\beta^{-1}(u) = u = u^{s^{-1}t}$. It follows that $s = t$ and $\beta = \gamma$. ■

Following is a function $\gamma \in Z_{\frac{1}{2}}(2)$, shown for the interval

$$\left[\left(\frac{1}{2}\right)^{2^3}, \left(\frac{1}{2}\right)^{2^{-3}}\right] = \bigcup_{n=-2}^{3} \left[\left(\frac{1}{2}\right)^{2^n}, \left(\frac{1}{2}\right)^{2^{n-1}}\right]$$



In summary, f and g generate strict t-norms with the same rth power for a given $r \in \mathbb{R}^+$ if and only if $f = s\gamma g$ for some $\gamma \in Z_{1/2}(r)$, $s \in \mathbb{R}^+$. From the construction, it is clear that $Z_{1/2}(r)$ has many elements that do not belong to $\mathbb{R}^+$, so there are many different strict t-norms that have the same rth power for a given r.

3 Powers of nilpotent t-norms

The powers of a general nilpotent t-norm depend on the powers of the Łukasiewicz t-norm, in the same sense that powers of strict t-norms depend on powers for multiplication. So we first consider the question of powers for the Łukasiewicz t-norm $x \bullet y = (x + y - 1) \vee 0$.

Lemma 12 *Let n be a positive integer and $x \bullet y = (x + y - 1) \vee 0$. Then*

$$\overbrace{x \bullet x \bullet \cdots \bullet x}^{n\ times} = (nx - n + 1) \vee 0.$$

Proof. We prove this by induction on n. The lemma clearly holds for $n = 1$. Assume that it holds for n. Then if $nx - n + 1 \geq 0$,

$$\begin{aligned}\overbrace{x \bullet x \bullet \cdots \bullet x}^{n+1 \text{ times}} &= x \bullet ((nx - n + 1) \vee 0) \\ &= (x + (nx - n + 1) - 1) \vee 0 \\ &= ((n+1)\,x - (n+1) + 1) \vee 0\end{aligned}$$

On the other hand, if $nx - n + 1 \leq 0$, then

$$\overbrace{x \bullet x \bullet \cdots \bullet x}^{n+1 \text{ times}} = x \bullet ((nx - n + 1) \vee 0) = x \bullet 0 = 0$$

and also

$$\begin{aligned}(n+1)\,x - (n+1) + 1 &= nx - n + 1 + (x - 1) \\ &\leq nx - n + 1 \leq 0\end{aligned}$$

so that

$$((n+1)\,x - (n+1) + 1) \vee 0 = 0$$

The lemma follows. ■

Now $x^{[n]} = (nx - n + 1) \vee 0$ is a continuous function mapping $[0, 1]$ onto $[0, 1]$, and restricts to an isomorphism $\left[\frac{n-1}{n}, 1\right] \approx [0, 1]$. Thus for each $x \in (0, 1]$ there is a unique solution to the equation $y^{[n]} = x$. Call this solution $x^{\left[\frac{1}{n}\right]}$.

Lemma 13 *Let n be a positive integer and $x \in (0, 1]$. Then $\frac{1}{n}x - \frac{1}{n} + 1$ is the unique solution to $y^{[n]} = x$.*

Proof. By Lemma 12,

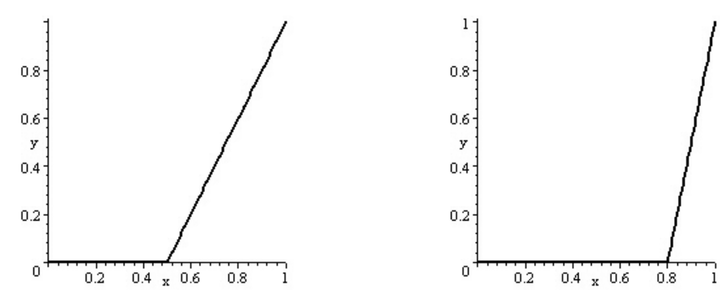
$$\left(\frac{1}{n}x - \frac{1}{n} + 1\right)^{[n]} = \left(n\left(\frac{1}{n}x - \frac{1}{n} + 1\right) - n + 1\right) \vee 0 = x \vee 0 = x$$

so $y = \frac{1}{n}x - \frac{1}{n} + 1$ is a solution to $y^{[n]} = x$. On the other hand, if $y^{[n]} = (ny - n + 1) \vee 0 = x \neq 0$, then $ny - n + 1 = x$ implies $y = \frac{1}{n}x - \frac{1}{n} + 1$. ■

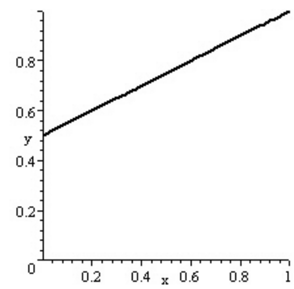
This observation leads to the following definition.

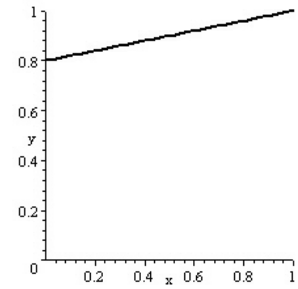
Definition 14 *For the Łukasiewicz t-norm, $x \in [0, 1]$, and positive real numbers r, the **rth power** of x is defined to be $x^{[r]} = (rx - r + 1) \vee 0$.*

Below are some examples of powers for the Łukasiewicz t-norm.



$x^{[2]} = (2x - 2 + 1) \vee 0$ and $x^{[5]} = (5x - 5 + 1) \vee 0$





$x^{[1/2]} = \left(\frac{1}{2}x - \frac{1}{2} + 1\right)$ and $x^{[1/5]} = \left(\frac{1}{5}x - \frac{1}{5} + 1\right)$

Proposition 15 *For positive real numbers r and s, and $x \in [0,1]$, the following hold.*

1. $\left(x^{[r]}\right)^{[s]} = x^{[rs]}$ *if* $x^{[r]} \neq 0$.
2. $x^{[r+s]} = x^{[r]} \bullet x^{[s]}$

Proof. Suppose $x^{[r]} \neq 0$. Then $rx - r + 1 > 0$, and

$$\begin{aligned} \left(x^{[r]}\right)^{[s]} &= (s\left((rx - r + 1) \vee 0\right) - s + 1) \vee 0 \\ &= (s\left(rx - r + 1\right) - s + 1) \vee 0 \\ &= (srx - sr + 1) \vee 0 = x^{[rs]} \end{aligned}$$

Suppose $x^{[r]} \neq 0$ and $x^{[s]} \neq 0$. Then $rx - r + 1 > 0$, and $sx - s + 1 > 0$. Thus

$$\begin{aligned} x^{[r]} \bullet x^{[s]} &= (((rx - r + 1) \vee 0) + ((sx - s + 1) \vee 0) - 1) \vee 0 \\ &= (rx - r + 1 + sx - s + 1 - 1) \vee 0 \\ &= ((r + s)\,x - (r + s) + 1) \vee 0 = x^{[r+s]} \end{aligned}$$

If $x^{[r]} = 0$ or $x^{[s]} = 0$ then $x^{[r]} \bullet x^{[s]} = 0$. Also, since either $rx - r + 1 \leq 0$ or $sx - s + 1 \leq 0$ and both $rx - r = r\,(x - 1) \leq 0$ and $sx - s = s\,(x - 1) \leq 0$, then

$$\begin{aligned} x^{[r+s]} &= ((r + s)\,x - (r + s) + 1) \vee 0 \\ &= (rx - r + sx - s + 1) \vee 0 = 0 \end{aligned}$$

Thus, in any case, $x^{[r+s]} = x^{[r]} \bullet x^{[s]}$. ■

From these propositions we see that for rationals $q = \frac{m}{n}$, $x^{\left[\frac{m}{n}\right]}$ is the mth power of the nth root of x and this definition depends only on the rational number $q = \frac{m}{n}$ and not on the particular representation of q as a quotient of integers.

For any positive integer n, 0 has multiple nth roots, namely the interval of numbers from 0 to the largest $x \in [0,1]$ satisfying $x^{[n]} = 0$. We make the convention that $0^{[n]} = 0$.

Again we see that the theory provides a mechanism for defining rth powers for $r \in \mathbb{R}^+$ in the nilpotent case. We use isomorphisms between nilpotent t-norms to extend the definition of powers to arbitrary nilpotent t-norms. If $f : (\mathbb{I}, \circ) \to (\mathbb{I}, \bullet)$ is an isomorphism of nilpotent t-norms—that is f is an automorphism of $\mathbb{I}$ satisfying $f\,(x \circ y) = f\,(x) \bullet f\,(y)$, then $f^{-1}\left((f\,(x))^{[r]}\right)$ is the corresponding rth power for $\circ$.

Definition 16 *Given a nilpotent t-norm $\circ$ and a positive real number r, the* ***rth power*** *of $\circ$ is defined to be the function $x^{[r]} = f^{-1}\left((r \cdot f\,(x) - r + 1) \vee 0\right)$, where f is the L-generator of $\circ$ and $r \cdot f(x)$ denotes the ordinary product of r and $f\,(x)$.*

The 2nd power of $\circ$ is $x^{[2]} = x \circ x$, commonly known as the **diagonal** of the t-norm $\circ$. Note that the diagonal of a nilpotent t-norm is a continuous nondecreasing function mapping $\mathbb{I}$ onto $\mathbb{I}$ that satisfies

1. $x \circ x < x$ for $x \in (0,1)$, and
2. there exists an $a \in (0,1)$ such that
   (a) $x \circ x = 0$ for $x \in [0,a]$, and
   (b) $x \circ x < y \circ y$ for $0 \leq x < y$ and $y \in (a,1)$.

Mesiar and Navara show in [7] that these are precisely the functions that occur as diagonals of nilpotent t-norms. The following theorem generalizes this to arbitrary powers of nilpotent t-norms.

Theorem 17 *Let δ be a nondecreasing function from $[0,1]$ to $[0,1]$, and let $r \in \mathbb{R}^+$. Then δ is the rth power for some nilpotent t-norm $\circ$ if and only if one of the following holds.*

1. *$r > 1$, $\delta(x) < x$ for all $x \in (0,1)$, and for some $a \in (0,1)$, δ is identically 0 on $[0,a]$ and induces an isomorphism $[a,1] \approx [0,1]$.*
2. *$r = 1$ and $\delta(x) = x$ for all $x \in (0,1)$.*
3. *$r < 1$, $\delta(x) > x$ for all $x \in [0,1)$, $\delta(0) = 0$ and for some $a \in (0,1)$, δ induces an isomorphism $(0,1] \approx (a,1]$.*

Proof. Suppose δ is the rth power for the nilpotent t-norm $\circ$ and $x \in (0,1)$. Let f be the isomorphism between $\circ$ and $\bullet$. Suppose $r > 1$. Then δ is identically 0 on $\left[0, f^{-1}\left(\frac{r-1}{r}\right)\right]$ and for $x > f^{-1}\left(\frac{r-1}{r}\right)$, $\delta(x) = f^{-1}(r \cdot f(x) - r + 1)$ which is a strictly increasing function from $\left[f^{-1}\left(\frac{r-1}{r}\right), 1\right]$ onto $[0,1]$. Let $x \in (0,1)$. If $\delta(x) = 0$, then $\delta(x) < x$. Otherwise, $\delta(x) = f^{-1}(r \cdot f(x) - r + 1)$. Now $f(x) < 1$ and $r - 1 > 0$ implies $(r-1) \cdot f(x) < r - 1$ which is equivalent to $r \cdot f(x) - r + 1 < f(x)$ or

$$\begin{aligned}\delta(x) &= f^{-1}((r \cdot f(x) - r + 1) \vee 0) \\ &= f^{-1}(r \cdot f(x) - r + 1) < f^{-1}(f(x)) = x.\end{aligned}$$

Suppose $r < 1$. Then $r \cdot f(x) - r + 1 \geq 0$ so that $\delta(x) = f^{-1}(r \cdot f(x) - r + 1)$. Moreover $0 < f(x) < 1$ and $0 < 1 - r < 1$ imply $f(x) \cdot (1 - r) < 1 - r$ which is equivalent to $r \cdot f(x) - r + 1 > f(x)$ or

$$\delta(x) = f^{-1}(r \cdot f(x) - r + 1) > f^{-1}(f(x)) = x.$$

Condition (2) is clear.

Now suppose for some $a \in (0,1)$, $\delta : [0,1] \to [0,1]$ is identically 0 on $[0,a]$, strictly increasing on $[a,1]$, and $\delta(x) < x$ for all $x \in (0,1)$. The sequence $\delta^{-n}(a)$, $n \in \mathbb{Z}^+$, is strictly increasing and $\lim_{n\to\infty} \delta^{-n}(a) = 1$. Let $r \in \mathbb{R}^+$, $r > 1$ and let g be any isomorphism $[0,a] \approx \left[0, \frac{r-1}{r}\right]$. For $0 \leq x \leq a$, define $f(x) = g(x)$. In general, define

$$f(x) = r^{-n} \cdot g(\delta^n(x)) - r^{-n} + 1$$

where $n \in \mathbb{Z}^+ \cup \{0\}$ is such that $0 < \delta^n(x) \leq a$. We need to show that

$$f(\delta(x)) = (r \cdot f(x) - r + 1) \vee 0$$

for all $x \in [0,1]$ and $f \in \mathrm{Aut}(\mathbb{I})$.

If $0 < \delta^n(x) \le a$, then $f(\delta(x)) = r^{-(n-1)} \cdot \left(g\left(\delta^{n-1}(\delta(x))\right)\right) - r^{-(n-1)} + 1$ while $f(x) = r^{-n} \cdot g(\delta^n(x)) - r^{-n} + 1$. Thus

$$\begin{aligned} r \cdot f(x) - r + 1 &= r \cdot \left(r^{-n} \cdot g(\delta^n(x)) - r^{-n} + 1\right) - r + 1 \\ &= r^{-(n-1)} \cdot g(\delta^n(x)) - r^{-(n-1)} + 1 \\ &= f(\delta(x)) \end{aligned}$$

Now f is continuous and increasing on each interval $\left[\delta^{-n}(a), \delta^{-(n+1)}(a)\right]$. At the endpoints we have

$$\begin{aligned} f\left(\delta^{-n}(a)\right) &= r^{-n} \cdot g(a) - r^{-n} + 1 \\ &= r^{-n} \cdot \left(\frac{r-1}{r}\right) - r^{-n} + 1 \\ &= 1 - r^{-(n+1)} \end{aligned}$$

and

$$\begin{aligned} f\left(\delta^{-n}(a)\right) = f\left(\delta\left(\delta^{-(n-1)}(a)\right)\right) &= \left(r \cdot f\left(\delta^{-(n-1)}(a)\right) - r + 1\right) \vee 0 \\ &= \left(r \cdot \left(r^{-(n-1)} \cdot g(a) - r^{-(n-1)} + 1\right) - r + 1\right) \vee 0 \\ &= \left(r \cdot \left(r^{-(n-1)} \cdot \frac{r-1}{r} - r^{-(n-1)} + 1\right) - r + 1\right) \vee 0 \\ &= \left(-r^{-n+1} + 1\right) \vee 0 = 1 - r^{-(n+1)} \end{aligned}$$

so f is continuous at the points $\delta^{-n}(a)$. It follows that $f \in \text{Aut}(\mathbb{I})$. Since $f(\delta(x)) = (r \cdot f(x) - r + 1) \vee 0$, δ is the rth power of the nilpotent t-norm defined by $x \circ y = f^{-1}(f(x) \bullet f(y))$.

The proof for condition (3) is similar, noting that $\lim_{n \to \infty} \delta^n(0) = 1$, letting g be any isomorphism $[0, \delta(0)] \approx [0, 1-r]$, and defining

$$f(x) = r^n \cdot g\left(\delta^{-n}(x)\right) - r^n + 1$$

where $n \in \mathbb{Z}^+ \cup \{0\}$ is such that $0 < \delta^{-n}(x) \le \delta(0)$. Condition (2) is trivial, since the identity function is the 1st power of all t-norms. ■

We now consider the question of uniqueness of the nilpotent t-norm having a given rth power. As for strict t-norms, there are many nilpotent t-norms that have the same rth power for any given power $r \neq 1$, but if *all* the rth powers are known, the nilpotent t-norm is uniquely determined. Again, this is a direct consequence of the proof of the representation theorem.

The following theorem reduces the problem of describing which nilpotent t-norms have the same rth power to the problem of describing the elements of the set

$$C(r) = \{f \in \text{Aut}(\mathbb{I}) : f((r \cdot x - r + 1) \vee 0) = (r \cdot f(x) - r + 1) \vee 0 \text{ for all } x \in [0,1]\}$$

Theorem 18 *Two nilpotent t-norms*

$$f^{-1}((f(x) + f(y) - 1) \vee 0) \text{ and } g^{-1}((g(x) + g(y) - 1) \vee 0)$$

have the same rth power if and only if fg^{-1} satisfies

$$fg^{-1}((r \cdot x - r + 1) \vee 0) = \left(\left(r \cdot f\left(g^{-1}(x)\right) - r + 1\right) \vee 0\right)$$

for all $x \in [0,1]$.

Proof. The two t-norms have the same rth power exactly when

$$g^{-1}\left(\left(r\cdot g\left(x\right)-r+1\right)\vee 0\right)=f^{-1}\left(\left(r\cdot f\left(x\right)-r+1\right)\vee 0\right)$$

Replacing x by $g^{-1}\left(x\right)$ gives

$$g^{-1}\left(\left(r\cdot x-r+1\right)\vee 0\right)=f^{-1}\left(\left(r\cdot fg^{-1}\left(x\right)-r+1\right)\vee 0\right)$$

from which the theorem follows. ■

In other words, f and g generate nilpotent t-norms with the same rth power if and only if $fg^{-1}\in C\left(r\right)$. It is straightforward to show that $C\left(r\right)$ is a subgroup of $\operatorname{Aut}\left(\mathbb{I}\right)$.

Theorem 19 *An automorphism f of $\mathbb{I}$ is in the group $C\left(r\right)$ for $r>1$ if and only if there is an automorphism $g\in\operatorname{Aut}\left(\left[0,\frac{1}{r}\right],\leq\right)$ such that*

$$f\left(x\right)=\frac{1}{r^{n}}\cdot g\left(r^{n}\cdot x-r^{n}+1\right)+1-\frac{1}{r^{n}}$$

for $1-\frac{1}{r^{n}}\leq x\leq 1-\frac{1}{r^{n+1}}$, $n\geq 0$. An automorphism f of $\mathbb{I}$ is in the group $C\left(r\right)$ for $r<1$ if and only if there is an automorphism $g\in\operatorname{Aut}\left(\left[0,r\right],\leq\right)$ such that

$$f\left(x\right)=r^{n}\cdot g\left(r^{-n}\cdot x-r^{-n}+1\right)+1-r^{n}$$

for $1-r^{n+1}\leq x\leq 1-r^{n}$, $n\geq 0$.

Proof. Let $r>1$. First suppose that $g\in\operatorname{Aut}\left(\left[0,\frac{1}{r}\right],\leq\right)$, f satisfies the equation stated in the theorem, and $f\left(1\right)=1$. To see that f is an automorphism of $\mathbb{I}$, first note that

$$[0,1)=\bigcup_{n=0}^{\infty}\left[1-\frac{1}{r^{n}},1-\frac{1}{r^{n+1}}\right]$$

so f is defined on $[0,1]$. The function g satisfies $g\left(0\right)=0$ and $g\left(1-\frac{1}{r}\right)=1-\frac{1}{r}$ and it follows easily that $f\left(1-\frac{1}{r^{n}}\right)=1-\frac{1}{r^{n}}$ for all $n\geq 0$. Since f is strictly increasing and continuous between these fixed points, we see that f is an automorphism of $[0,1]$.

Let $x\in[0,1]$ with $1-\frac{1}{r^{n}}\leq x\leq 1-\frac{1}{r^{n+1}}$ for some $n\geq 0$. We need to show that

$$f\left(\left(rx-r+1\right)\vee 0\right)=\left(rf\left(x\right)-r+1\right)\vee 0$$

If $n=0$, then $rx-r+1$ and $rf\left(x\right)-r+1$ are both less than or equal to $r\left(1-\frac{1}{r}\right)-r+1=0$. Thus $f\left(\left(r\cdot x-r+1\right)\vee 0\right)=f\left(0\right)=g\left(0\right)=0$ and $\left(r\cdot f\left(x\right)-r+1\right)\vee 0=0$. If $n>0$, then

$$1-\frac{1}{r^{n-1}}\leq rx-r+1\leq 1-\frac{1}{r^{n}}$$

Thus

$$\begin{aligned}f\left(r\cdot x-r+1\right)&=\frac{1}{r^{n-1}}\cdot g\left(r^{n-1}\cdot\left(r\cdot x-r+1\right)-r^{n-1}+1\right)+1-\frac{1}{r^{n-1}}\\&=\frac{1}{r^{n-1}}\cdot g\left(r^{n}\cdot x-r^{n}+1\right)+1-\frac{1}{r^{n-1}}\end{aligned}$$

and

$$\begin{aligned}r\cdot f\left(x\right)-r+1&=r\cdot\left(\frac{1}{r^{n}}\cdot g\left(r^{n}\cdot x-r^{n}+1\right)+1-\frac{1}{r^{n}}\right)-r+1\\&=\frac{1}{r^{n-1}}\cdot g\left(r^{n}\cdot x-r^{n}+1\right)+1-\frac{1}{r^{n-1}}\end{aligned}$$

Thus $f \in C(r)$.

Now suppose $f \in C(r)$. Then $0 \leq x \leq 1 - \frac{1}{r}$ if and only if $r \cdot x - r + 1 \leq 0$ if and only if $f((r \cdot x - r + 1) \vee 0) = f(0) = 0$, in which case $r \cdot f(x) - r + 1 \leq 0$. It follows that $f(x) \leq 1 - \frac{1}{r}$ on $\left[0, \frac{r-1}{r}\right]$. For $r \cdot x - r + 1 > 0$, we have $r \cdot f(x) - r + 1 = f(r \cdot x - r + 1) > 0$, so that $f(x) \geq 1 - \frac{1}{r}$. Since f is a continuous function, $f\left(1 - \frac{1}{r}\right) = 1 - \frac{1}{r}$, so f induces automorphisms $\left[0, 1 - \frac{1}{r}\right] \approx \left[0, 1 - \frac{1}{r}\right]$ and $\left[1 - \frac{1}{r}, 1\right] \approx \left[1 - \frac{1}{r}, 1\right]$. Let g be the restriction of f to $\left[0, 1 - \frac{1}{r}\right]$. Then for $1 - \frac{1}{r^0} \leq x \leq 1 - \frac{1}{r^{0+1}}$, $f(x) = g(x) = \frac{1}{r^0}\left(g\left(r^0 \cdot x - r^0 + 1\right) + r^0 - 1\right)$. We will induct on n.

Suppose $n \geq 0$ and for $1 - \frac{1}{r^n} \leq x \leq 1 - \frac{1}{r^{n+1}}$, $f(x) = \frac{1}{r^n}\left(g\left(r^n \cdot x - r^n + 1\right) + r^n - 1\right)$. Then, for $1 - \frac{1}{r^{n+1}} \leq x \leq 1 - \frac{1}{r^{n+2}}$, we have $1 - \frac{1}{r^n} \leq r \cdot x - r + 1 \leq 1 - \frac{1}{r^{n+1}}$, so that

$$\begin{aligned} f(r \cdot x - r + 1) &= \frac{1}{r^n}\left(g\left(r^n \cdot (r \cdot x - r + 1) - r^n + 1\right) + r^n - 1\right) \\ &= \frac{1}{r^n} g\left(r^{n+1} x - r^{n+1} + 1\right) + 1 - \frac{1}{r^n} \end{aligned}$$

But on the other hand, since $f \in C(r)$,

$$f(r \cdot x - r + 1) = r \cdot f(x) - r + 1$$

so that

$$r \cdot f(x) - r + 1 = \frac{1}{r^n} \cdot g\left(r^{n+1} x - r^{n+1} + 1\right) + 1 - \frac{1}{r^n}$$

whence

$$\begin{aligned} f(x) &= \frac{1}{r} \cdot \left(\frac{1}{r^n} \cdot g\left(r^{n+1} x - r^{n+1} + 1\right) + 1 - \frac{1}{r^n} + r - 1\right) \\ &= \frac{1}{r^{n+1}} \cdot g\left(r^{n+1} x - r^{n+1} + 1\right) + 1 - \frac{1}{r^{n+1}} \end{aligned}$$

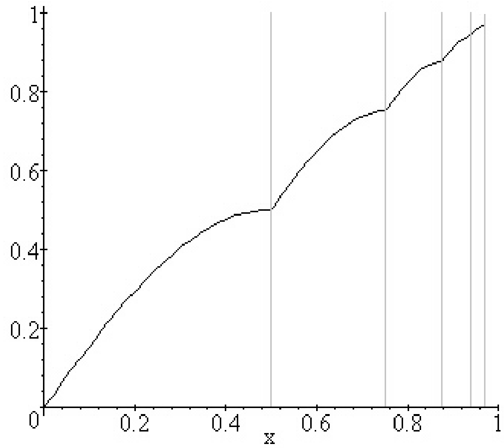
The claim follows by induction on n.

The proof for $r < 1$ is similar, replacing $1 - \frac{1}{r}$ by r throughout. ■

Corollary 20 *For $r > 1$, the group $C(r)$ is isomorphic to* $\mathrm{Aut}\left(\left[0, 1 - \frac{1}{r}\right], \leq\right)$ *via the restriction $f \mapsto f \restriction \left[0, 1 - \frac{1}{r}\right]$. For $r < 1$, the group $C(r)$ is isomorphic to* $\mathrm{Aut}([0, r], \leq)$ *via the restriction $f \mapsto f \restriction [0, r]$. In particular, $C(r)$ is isomorphic to* $\mathrm{Aut}(\mathbb{I})$.

Following is a function $f \in C_2$, shown for the interval

$$\left[0, 1 - 2^{-4}\right] = \left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right] \cup \left[\frac{3}{4}, \frac{7}{8}\right] \cup \left[\frac{7}{8}, \frac{15}{16}\right]$$



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In summary, f and g generate nilpotent t-norms with the same rth power for a given $r \in \mathbb{R}^+$ if and only if $f = \gamma g$ for some $\gamma \in C(r) \approx \mathrm{Aut}(\mathbb{I})$. Thus there are uncountably many different nilpotent t-norms that have the same rth power for a given r.

4 Powers of continuous t-norms

An arbitrary continuous t-norm is an ordinal sum of countably many strict, nilpotent, and idempotent t-norms. Continuous t-norms were characterized by Ling [6]. We rephrase this characterization to fit the setting of this paper.

Theorem 21 *A binary operation* $\circ : [0,1]^2 \to [0,1]$ *is a continuous t-norm if and only if there is a disjoint collection of open subintervals* $\{(a_k, b_k)\}_{k \in K}$ *of* $[0,1]$ *and a family of isomorphisms* $\{f_k : ([a_k, b_k], \leq) \to ([0,1], \leq)\}_{k \in K}$*, such that for each* k*,*

$$x \circ y = \begin{cases} f_k^{-1}(f_k(x) f_k(y)) & \text{if } x, y \in [a_k, b_k] \text{ and } x \circ x \\ & \text{is strictly increasing on } [a_k, b_k] \\ f_k^{-1}((f_k(x) + f_k(y) - 1) \vee 0) & \text{if } x, y \in [a_k, b_k] \text{ and } x \circ x \\ & \text{is not strictly increasing on } [a_k, b_k] \\ x \wedge y & \text{otherwise} \end{cases}$$

For continuous t-norms, we define rth powers for positive real numbers r as follows.

Definition 22 *Given a continuous t-norm* $\circ$*, let* $\{f_k : ([a_k, b_k], \leq) \to ([0,1], \leq)\}_{k \in K}$ *be the family of isomorphisms described in Theorem 21. Let* $r \in \mathbb{R}^+$*. The* r*th power of* $\circ$ *is defined by*

$$x^{[r]} = \begin{cases} f_k^{-1}((f_k(x))^r) & \text{if } x \in [a_k, b_k] \text{ and } x \circ x \\ & \text{is strictly increasing on } [a_k, b_k] \\ f_k^{-1}((r \cdot f_k(x) - r + 1) \vee 0) & \text{if } x \in (a_k, b_k] \text{ and } x \circ x \\ & \text{is not strictly increasing on } [a_k, b_k] \\ x & \text{otherwise} \end{cases}$$

For positive integers n this definition satisfies

$$x^{[n]} = \overbrace{x \circ x \circ \cdots \circ x}^{n \text{ times}}$$

and $x^{\left[\frac{1}{n}\right]}$ is the unique smallest solution to $\left(x^{\left[\frac{1}{n}\right]}\right)^{[n]} = x$. The diagonals ($r = 2$) are exactly the nondecreasing functions δ from $[0,1]$ onto $[0,1]$ such that $\delta(x) \leq x$ for all x and satisfying $\delta(\delta(x)) = \delta(x)$ whenever $\delta(x) = \delta(y)$ for $y \neq x$ ([7], Theorem 4.3). The characterization of rth powers of continuous t-norms is similar.

Theorem 23 *Let $r \in \mathbb{R}^+$. A nondecreasing left-continuous function $\delta : [0,1] \to [0,1]$ is the rth power for some continuous t-norm $\circ$ if and only if one of the following holds:*

1. *$r > 1$ and $\delta(x) \leq x$ for all $x \in [0,1]$, δ is surjective, and $\delta(\delta(x)) = \delta(x)$ whenever $\delta(x) = \delta(y)$ for $y \neq x$.*
2. *$r = 1$ and $\delta(x) = x$ for all $x \in [0,1]$.*
3. *$r < 1$ and $\delta(x) \geq x$ for all $x \in [0,1]$, δ is injective, and $\delta(\delta(x)) = \delta(x)$ whenever $\lim_{y \to x^+} \delta(y) \neq \delta(x)$.*

Proof. Suppose δ is the rth power for some continuous t-norm, with the associated disjoint collection of open subintervals $\{(a_k, b_k)\}_{k \in K}$ of $[0,1]$ and family of isomorphisms $\{f_k : ([a_k, b_k], \leq) \to ([0,1], \leq)\}_{k \in K}$ satisfying the conditions of Theorem 21. It is clear that δ is nondecreasing and that $\delta(x) = x$ if and only if $x \notin \cup\{(a_k, b_k)\}_{k \in K}$. Suppose $\delta(x) = \delta(y)$ for $x < y$. If $\delta(x) = x$, then certainly $\delta(\delta(x)) = \delta(x)$, so suppose $\delta(x) \neq x$ whence $x \in (a_k, b_k)$ for some k and $a_k = \delta(a_k) \leq \delta(x) \leq \delta(y) \leq \delta(b_k) = b_k$. Since δ is strictly increasing on $[a_k, b_k]$ for some $a_k \leq c_k < b_k$ and $\delta(w) = \delta(a_k)$ for $w \in [a_k, c_k]$ it must be that $a_k = \delta(a_k) = \delta(x) = \delta(y) < \delta(b_k) = b_k$. Thus in this case also, $\delta(\delta(x)) = \delta(x)$. The three conditions for $r > 1$, $r = 1$, and $r < 1$ now follow directly from the corresponding theorems for strict and nilpotent t-norms.

Assume $r > 1$ and condition (1) holds. Let δ be a function satisfying the conditions of the theorem and let $D = \{x \in [0,1] : \delta(x) = x\}$. Since δ is nondecreasing, left continuous, and onto, D is a closed set, so $[0,1] \setminus D$ is a countable union $\cup\{(a_k, b_k)\}_{k \in K}$ of open intervals. For each k, define $\varphi_k(x) = (x - a_k)/(b_k - a_k)$. Then $\varphi_k : [a_k, b_k] \to [0,1]$ is an order isomorphism. Suppose δ is strictly increasing on $[a_k, b_k]$. Then δ is an automorphism of $([a_k, b_k], \leq)$ and $\delta_k = \varphi_k \delta \varphi_k^{-1}$ is an automorphism of $[0,1]$ satisfying $\delta_k(x) < x$ for all $x \in (0,1)$. Thus by Theorem 7 $\delta_k = g_k^{-1} r g_k$ for some $g_k \in \mathrm{Aut}(\mathbb{I})$, and $\delta = \varphi_k^{-1} \delta_k \varphi_k = \varphi_k^{-1} g_k^{-1} r g_k \varphi_k = f_k^{-1} r f_k$ on $[a_k, b_k]$ where $f_k = g_k \varphi_k$.

Now suppose δ is not strictly increasing on $[a_k, b_k]$. Thus there exists $a_k \leq x < y \leq b_k$ with $\delta(x) = \delta(y)$. By hypothesis, $\delta(\delta(x)) = \delta(x)$ which implies that $\delta(x) \notin \cup\{(a_k, b_k)\}_{k \in K}$. But $a_k = \delta(a_k) \leq \delta(x) \leq x < b_k$ so we must have $\delta(x) = \delta(a_k) = a_k$. It follows that for some $w \in (a_k, b_k)$, δ is identically a_k on $[a_k, w]$ and strictly increasing on $[w, b_k]$. Then $\delta_k = \varphi_k \delta \varphi_k^{-1}$ satisfies the hypothesis of Theorem 17 and there is an automorphism g_k of $[0,1]$ such that $\delta_k(x) = g_k^{-1}((r \cdot g_k(x) - r + 1) \vee 0)$. Then for $f_k = g_k \varphi_k$, we have $\delta(x) = f_k^{-1}((r \cdot f_k(x) - r + 1) \vee 0)$ for $x \in [a_k, b_k]$. By Theorem 21, the disjoint collection of open subintervals $\{(a_k, b_k)\}_{k \in K}$ of $[0,1]$ with the family of isomorphisms $\{f_k : ([a_k, b_k], \leq) \to ([0,1], \leq)\}_{k \in K}$, determine a continuous t-norm that has δ as rth power.

Now assume $r < 1$ and let δ be a function satisfying condition (3). Let $D = \{x \in [0,1] : \delta(x) = x\}$. Since δ is nondecreasing and left continuous and $\delta(x) \geq x$ for all x, D is a closed set. Thus $[0,1] \setminus D$ is a countable union $\cup\{(a_k, b_k)\}_{k \in K}$ of open intervals. For each k, define $\varphi_k(x) = (x - a_k)/(b_k - a_k)$. Then $\varphi_k : [a_k, b_k] \to [0,1]$ is an order isomorphism.

Suppose δ maps $[a_k, b_k]$ onto $[a_k, b_k]$. Then δ induces an automorphism of $([a_k, b_k], \leq)$ and $\delta_k = \varphi_k \delta \varphi_k^{-1}$ is an automorphism of $[0,1]$ satisfying $\delta_k(x) < x$ for all $x \in (0,1)$. Thus by Theorem 7 $\delta_k = g_k^{-1} r g_k$ for some $g_k \in \mathrm{Aut}(\mathbb{I})$, and $\delta = \varphi_k^{-1} \delta_k \varphi_k = \varphi_k^{-1} g_k^{-1} r g_k \varphi_k = f_k^{-1} r f_k$ on $[a_k, b_k]$ where $f_k = g_k \varphi_k$.

Now suppose δ does not map $[a_k, b_k]$ onto $[a_k, b_k]$. Thus there exists $a_k \leq x < b_k$ with $\lim_{y \to x^+} \delta(y) \neq \delta(x)$. By hypothesis, $\delta(\delta(x)) = \delta(x)$ which implies that $\delta(x) \notin \cup \{(a_k, b_k)\}_{k \in K}$. But $a_k = \delta(a_k) \leq \delta(x) \leq x < b_k$ so we must have $\delta(x) = \delta(a_k) = a_k$. Since δ is one-to-one, it follows that $x = a_k$, and δ induces an isomorphism $(a_k, b_k] \approx (a, b_k]$ for $a = \lim_{y \to x^+} \delta(y)$. Then $\delta_k = \varphi_k \delta \varphi_k^{-1}$ satisfies the hypothesis of Theorem 17 and there is an automorphism g_k of $[0,1]$ such that $\delta_k(x) = g_k^{-1}((r \cdot g_k(x) - r + 1) \vee 0)$. Then for $f_k = g_k \varphi_k$, we have $\delta(x) = f_k^{-1}((r \cdot f_k(x) - r + 1) \vee 0)$ for $x \in [a_k, b_k]$.

By Theorem 21, the disjoint collection of open subintervals $\{(a_k, b_k)\}_{k \in K}$ of $[0,1]$ with the family of isomorphisms $\{f_k : ([a_k, b_k], \leq) \to ([0,1], \leq)\}_{k \in K}$ determines a continuous t-norm that has δ as rth power. ■

The characterization of all continuous t-norms with rth power δ can be pieced together from the characterizations for strict and nilpotent t-norms. The explicit description of all continuous t-norms with a given rth power is sufficiently complicated that we omit stating it here.

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