Projective Classes of Completely Decomposable Abelian Groups

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1. Introduction The relative homological algebras generated by taking classes of completely decomposable groups as projectives have several interesting properties which we investigate in this paper. A group[1] is completely decomposable if it has a decomposition as a direct sum of rank one groups, the rank one groups being the group Q of rationals, groups of the type $Z(p^\infty)$ and subgroups of Q and the $Z(p^\infty)$'s.

If $\mathcal{C}$ is a class of groups, the relative homological algebra generated by $\mathcal{C}$ consists of the class $\mathcal{E} = \mathcal{E}(\mathcal{C})$ of all short exact sequences $A \rightarrowtail B \twoheadrightarrow C$ of Abelian groups for which the sequence

$$0 \longrightarrow \operatorname{Hom}(G, A) \longrightarrow \operatorname{Hom}(G, B) \longrightarrow \operatorname{Hom}(G, C) \longrightarrow 0 \qquad (*)$$

is exact for every $G \in \mathcal{C}$. Associated with $\mathcal{E}$ are the class $\overline{\mathcal{C}}$ of all groups G for which (*) is exact for all sequences $A \rightarrowtail B \twoheadrightarrow C \in \mathcal{E}$, and the class $\mathcal{I} = \mathcal{I}(\mathcal{E})$ of all groups G for which the sequence

$$0 \longrightarrow \operatorname{Hom}(C, G) \longrightarrow \operatorname{Hom}(B, G) \longrightarrow \operatorname{Hom}(A, G) \longrightarrow 0$$

is exact for every $A \rightarrowtail B \twoheadrightarrow C \in \mathcal{E}$. The class $\mathcal{E}$ is the class of proper short exact sequences. The class $\overline{\mathcal{C}}$ is the class of projectives and the projective closure of $\mathcal{C}$. The class $\mathcal{I}$ is the class of injectives.

If $\mathcal{C}$ is any set of rank one groups it follows immediately from Theorem 1 in [6] that there are enough projectives, i.e. for each group A there is a proper exact sequence

$$K \rightarrowtail C \twoheadrightarrow A$$

with $C \in \overline{\mathcal{C}}$. It also follows from this theorem that $\overline{\mathcal{C}}$ consists of all summands of direct sums of groups in $\mathcal{C} \cup \{Z\}$, where Z is the group of integers. It then follows

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[1]The word group will mean Abelian group throughout this paper.

from theorems of Kulikov [4], Kaplansky [3], and Baer [1], and theorems about completely decomposable torsion groups that $\overline{\mathcal{C}}$ consists of all direct sums of groups in $\mathcal{C} \cup \{Z\}$. Kulikov proved that summands of countable completely decomposable torsion free groups were again completely decomposable, and Kaplansky reduced the general prohlem to the countable case. Baer proved that any two decompositions of a completely decomposable torsion free group into a direct sum of rank one groups are isomorphic, so $\overline{\mathcal{C}}$ does not contain rank one torsion free groups outside of $\overline{\mathcal{C}} \cup \{Z\}$. Similar theorems for completely decomposable torsion groups are well known.

Thus the classes of projectives in these relative homological algebras are completely known. The aspects we will explore are descriptions of the proper exact sequences, and determinations of the relative homological dimensions.

A long exact sequence

$$\ldots \to G_{n+1} \xrightarrow{f_n} G_n \xrightarrow{f_{n-1}} G_{n-1} \to \ldots$$

is a proper exact sequence if each short exact sequence

$$\operatorname{Im} f_n \rightarrowtail G_n \twoheadrightarrow \operatorname{Im} f_{n-1}$$

is a proper short exact sequence. A proper projective resolution of a group G is a proper exact seqence

$$\ldots \to P_{n+1} \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \twoheadrightarrow G \qquad (**)$$

with each $P_i \in \overline{\mathcal{C}}$. It follows easily from the existence of enough projectives that every group has a proper projective resolution. If G has a proper projective resolution (**) with $P_m = 0$ for $m > n$, G has $\mathcal{C}$-projective dimension $\leq n$. If G has no such resolution for any integer n, G has inønite $\mathcal{C}$-projective dimension. The projective dimension of the relative homological algebra is the least upper bound of the $\mathcal{C}$-projective dimensions of all groups. We can ascertain dimensions only in a few special cases. Also we can determine the class of injectives only in a few special cases.

2. Classes of divisible and free groups. Projective classes which contain divisible groups were investigated in [6]. It was proved there that the group Q of rationals is in $\overline{\mathcal{C}}$ if and only if for each short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{E} = \mathcal{E}(\mathcal{C})$, the sequence of subgroups

$$A \cap dB \rightarrowtail dB \twoheadrightarrow dC$$

is exact (where dG denotes the maximum divisible subgroup of G) and $A \cap dB$ is the direct sum of a cotorsion group and a divisible group (i.e. $\operatorname{Ext}(Q, A \cap dB) = 0$). The group $Z(p^\infty)$ is in $\overline{\mathcal{C}}$ if and only if for each short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{E} = \mathcal{E}(\mathcal{C})$ the sequence of subgroups

$$dA_p \rightarrowtail dB_p \twoheadrightarrow dC_p$$

is exact (where dG_p is the maximal divisible p-primary subgroup of G). Also, $\overline{\mathcal{C}}$ contains all divisible groups if and only if for each short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{E} = \mathcal{E}(\mathcal{C})$ the sequence of subgroups

$$dA \rightarrowtail dB \twoheadrightarrow dC$$

is exact. These theorems characterize the proper exact sequences in all cases where $\mathcal{C}$ is a set of divisible groups.

If $\mathcal{C}$ is any set of rank one torsion divisible groups or the set of all rank one divisible groups then the class of divisible groups in $\overline{\mathcal{C}}$ is closed under homomorphic images. Thus each group G has a unique maximum divisible subgroup $\mathcal{C}(G)$ belonging to $\overline{\mathcal{C}}$, namely, $\mathrm{Im}\left(\sum_{C \in C} \sum_{\mathrm{Hom}(C,G)} C \longrightarrow G\right)$.

Let $F \twoheadrightarrow G$ be a free group mapping onto G and let $\mathcal{C}(G) \to G$ be the inclusion map. It is easy to see that the sum of these two maps

$$K \rightarrowtail F \oplus \mathcal{C}(G) \twoheadrightarrow G$$

gives a proper projective resolution of length ≤ 1 The group K must be a free group, since projecting into F is a monomorphism.

The only other possibilities are sets $\mathcal{C}$ of divisible rank one groups with $Q \in \mathcal{C}$ and $Z(p^\infty) \notin \mathcal{C}$ for some prime p. We show that in this case $Z(p^\infty)$ has $\mathcal{C}$-projective dimension 2, and that the relative homological algebra generated by $\mathcal{C}$ has projective dimension 2.

If G is any group, there is a decomposition $G = D \oplus R$ with D divisible and R reduced (i.e. R contains no non-zero divisible subgroup). Then D decomposes as $D = D_0 \oplus D_1$ with $D_0 \in \overline{\mathcal{C}}$ and D_1 containing no non-zero subgroups in $\overline{\mathcal{C}}$. If we have a proper projective resolution of D_1

$$\ldots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \twoheadrightarrow D_1$$

we may assume each P_i is torsion free, since, for example, $\mathrm{Hom}(Z(p^\infty), D_1) = 0$ for all $Z(p^\infty) \in \mathcal{C}$. Let $F_1 \rightarrowtail F_0 \twoheadrightarrow R$ be a free resolution of R and let $D_0 \to D_0$ be the identity map. It is easy to see that the sum of these sequences

$$\ldots \to P_3 \to P_2 \to P_1 \oplus F_1 \to D_0 \oplus P_0 \oplus F_0 \twoheadrightarrow D_0 \oplus D_1 \oplus R = G$$

is a proper projective resolution of G. This reduces the problems of relative homological dimension to determining what happens for divisible groups D which have no nonzero subgroups in $\mathcal{C}$, and this quickly reduces to ønding resolutions for groups $Z(p^\infty)$ which do not belong to $\mathcal{C}$.

Suppose $Z(p^\infty) \notin \mathcal{C}$ and let P be the additive group of p-adic integers. Then $P \otimes Q$ is torsion-free divisible and there is an exact sequence

$$P \rightarrowtail P \otimes Q \twoheadrightarrow Z(p^\infty)$$

This sequence is proper since P is cotorsion. Let $K \rightarrowtail F \twoheadrightarrow P$ be a free resolution of P. This is a proper projective resolution, since P is reduced. Composing the two sequences gives a proper projective resolution

$$K \rightarrowtail F \to P \otimes Q \twoheadrightarrow Z\left(p^{\infty}\right)$$

of $Z\left(p^{\infty}\right)$. Thus $Z\left(p^{\infty}\right)$ has a $\mathcal{C}$-projective dimension at most 2. Let

$$F_1 \oplus D_1 \rightarrowtail F_0 \oplus D_0 \twoheadrightarrow Z\left(p^{\infty}\right)$$

be a short exact sequence with F_1, F_0 free and D_1, D_0 torsion free divisible. If the sequence is proper, then so is

$$F_1 \rightarrowtail F_0 \oplus (D_0/D_1) \twoheadrightarrow Z\left(p^{\infty}\right)$$

so we may as well assume $D_1 = 0$. Then if the sequence is proper, $F_1 \cap D_0$ is both free and cotorsion plus divisible. This implies $F_1 \cap D_0 = 0$. Thus the map $D_0 \to Z\left(p^{\infty}\right)$ is a monomorphism. But this implies $D_0 = 0$ since D_0 is torsion free. But clearly the sequence $F_1 \rightarrowtail F_0 \twoheadrightarrow Z\left(p^{\infty}\right)$ is not proper. We conclude the original sequence is not proper. Thus $Z\left(p^{\infty}\right)$ has dimension exactly 2 if $\mathcal{C}$ is a set of rank one divisible groups with $Q \in \mathcal{C}$ and $Z\left(p^{\infty}\right) \notin \mathcal{C}$, and the relative homological algebra has dimension 2.

The relative Ext functors can be computed in terms of Hom and Ext for $\mathcal{C}$ any set of divisible groups. If $Q \notin \overline{\mathcal{C}}$ then any group G has a maximum divisible subgroup D belonging to $\overline{\mathcal{C}}$, and it is easy to show that

$$\mathcal{E}^1(G, A) \cong \mathcal{E}^1(G/D, A) = \operatorname{Ext}(G/D, A)$$

and $\mathcal{E}^n(G, A) = 0$ for $n > 1$. Now suppose $Q \in \overline{\mathcal{C}}$ and let $I = \{p : Z\left(p^{\infty}\right) \notin \overline{\mathcal{C}}\}$. A group G has a decomposition $G = D_1 \oplus D_2 \oplus R$ with $D_1 \in \overline{\mathcal{C}}$, D_2 a divisible subgroup of G which has no non-zero subgroups in $\overline{\mathcal{C}}$, and R reduced. Let r_p be the p-rank of D_2. Then for any group A, we have $\mathcal{E}^n(G, A) \cong \mathcal{E}^n(D_2, A) \oplus \mathcal{E}^n(R, A)$. Also, $\mathcal{E}^1(R, A) = \operatorname{Ext}(R, A) \cong \operatorname{Ext}(G/dG, A)$, and $\mathcal{E}^n(R, A) = 0$ for $n > 1$. The problem comes down to computing $\mathcal{E}^n\left(Z\left(p^{\infty}\right), A\right)$ for $p \in I$, $n = 1, 2$. For each prime $p \in I$, there is a proper exact sequence

$$P(p) \rightarrowtail D \twoheadrightarrow Z\left(p^{\infty}\right)$$

with $P(p)$ the additive group of p-adic integers and D torsion free divisible. This leads to the exact sequence

$$\begin{aligned} &\operatorname{Hom}(D, A/dA) = 0 \to \operatorname{Hom}(P(p), A/dA) \to \mathcal{E}^1(Z\left(p^{\infty}\right), A/dA) \to \\ &\to \mathcal{E}^1(D, A/dA) = 0 \to \mathcal{E}^1(P(p), A/dA) \to \mathcal{E}^2(Z\left(p^{\infty}\right), A/dA) \to \\ &\to \mathcal{E}^2(D, A/dA) = 0 \end{aligned}$$

and we have isomorphisms

$$\operatorname{Hom}(P(p), A/dA) \cong \mathcal{E}^1(Z\left(p^{\infty}\right), A/dA) \cong \mathcal{E}^1(Z\left(p^{\infty}\right), A)$$

and

$$\operatorname{Ext}(P(p), A) \cong \mathcal{E}^1(P(p), A/dA) \cong \mathcal{E}^2(Z(p^\infty), A/dA) \cong \mathcal{E}^2(Z(p^\infty), A).$$

Combining these results, we have

$$\mathcal{E}^1(G, A) \cong \operatorname{Ext}(G/dG, A) \oplus \prod_{p \in I} \prod_{r_p} \operatorname{Hom}(P(p), A/dA)$$

and

$$\mathcal{E}^2(G, A) \cong \prod_{p \in I} \prod_{r_p} \operatorname{Ext}(P(p), A)$$

whenever $Q \in \overline{\mathcal{C}}$.

The other aspect of these relative homological algebras is the class of injectives. In this aspect, the relative homological algebras are deøcient. If G is any reduced group there is a prime p for which $pG \neq G$. For such a prime, there is a non-splitting extension $G \rightarrowtail H \twoheadrightarrow Z(p)$. But such an extension is proper for any class $\mathcal{C}$ of rank one divisible groups. Thus there are no reduced injectives for any of these relative homological algebras.

3. Completely decomposable groups. We turn now to the set $\mathcal{C}$ of all rank one groups. As mentioned in the introduction, $\overline{\mathcal{C}}$ is the class of all completely decomposable groups. A couple of the properties of the proper exact sequences in $\mathcal{E} = \mathcal{E}(\mathcal{C})$ are easily observed. First $\mathcal{C}$ contains all primary cyclic groups, so the sequences in $\mathcal{E}$ are pure, i.e. if $A \subset B$ then $A \rightarrowtail B \twoheadrightarrow B/A \in \mathcal{E}$ implies $A \cap nB = nA$ for all positive integers n. Second, $\mathcal{C}$ contains all rank one divisible groups so the sequcnee $dA \rightarrowtail dB \twoheadrightarrow dC$ of maximum divisible subgroups is exact whenever $A \rightarrowtail B \twoheadrightarrow C$ belongs to $\mathcal{E}$. In the case both A and C are torsion, these two conditions are suŒcient to imply an extension $A \rightarrowtail B \twoheadrightarrow C$ is proper.

Theorem 1 If A, C are torsion, then a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ is proper with respect to the projective class of completely decomposable groups if and only if the sequence is pure exact and the sequence $dA \rightarrowtail dB \twoheadrightarrow dC$ of maximum divisible subgroups is exact.

Proof. We have already observed the necessity of the conditions. Suppose $A \rightarrowtail B \twoheadrightarrow C$ is pure exact and the sequence of maximum divisible subgroups is exact. Then $\operatorname{Hom}(G, B) \to \operatorname{Hom}(G, C)$ is an epimorphism for all groups $G \in \mathcal{C}$ which are either torsion or divisible. If $Z \subseteq G \subseteq Q$ and $f : G \to C$ then the image of f is torsion, hence isomorphic to a subgroup of Q/Z. Thus $\operatorname{Im} f \in \overline{\mathcal{C}}$, in fact $\operatorname{Im} f = \sum T_p$ where $T_p \cong Z(p^n)$ for some $0 \leq n \leq \infty$. The inclusion $\operatorname{Im} f \subseteq C$, can thus be factored through $B \to C$. But this factors f as well, so we have $\operatorname{Hom}(G, B) \to \operatorname{Hom}(G, C)$ an epimorphism for all $G \in \mathcal{C}$, and the sequence is proper. ■

In the case both A and C are torsion free, the proper sequences have been characterized by Lyapin [5]. Lyapin's theorem and other relevant information are in [2], 46. We need a couple of deønitions before stating the theorem. If G is a torsion free group and $x \in G$, the height of x in G is the sequence $H_G(x) = \langle k_1, k_2, \ldots \rangle$ where, if $p_1, p_2, \ldots$ is a listing of the primes in their natural order, $k_i = k$ if $x \in p_i^k G$ and $x \notin p_i^{k+1} G$, and $k_i = \infty$ if $x \in p_i^k G$ for all positive integers k. If G and G/H are both torsion free then a coset $x \in G/H$ is regular if x contains an element a such that $H(b) \leq H(a)$ for all $b \in x$.

Theorem (Lyapin). Let the factor group G/H of the torsion free group G be a completely decomposable torsion free group. Then G is the direct sum of H and a completely decomposable group if and only if every coset of G/H is regular.

Our second theorem follows as an immediate corollary.

Theorem 2 If A, C are torsion free, then a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ is proper with respect to the projective class of completely decomposable groups if and only if every coset of B mod A is regular (i.e. contains an element of maximal height).

In order to state a general theorem we need to modify the deønition of regular coset. If $x \in G$, $o(x)$ denotes the smallest positive integer n such that $nx = 0$, if such exists. Otherwise $o(x) = \infty$. If α is an ordinal, $p^\alpha G$ is deøned inductively as follows. If α is a limit ordinal and $p^\beta G$ is deøned for $\beta < \alpha$, then $p^\alpha G = \cap_{\beta<\alpha} p^\beta G$. If $\alpha = \beta + 1$ and $p^\beta G$ is deøned, then $p^\alpha G = p(p^\beta G)$. Now if $x \in G$, $H_G(x) = \langle k_1, k_2, \ldots \rangle$ where k_i is the ordinal such that at $x \in p_i^k G$ and $x \notin p_i^{k+1} G$, if such exists, and $k_i = \infty$ if $x \in p_i^k G$ for all ordinals k. We will encounter only heights with the k_i's either integers or ∞. Note that to say x has height ∞ at the prime p means x belongs to the maximum p-divisible subgroup of G. If x also has order a power of the prime p, this means x belongs to the p-primary divisible subgroup dG of G.

Deønition. Let H be a subgroup of a group G. A coset $x \in G/H$ is regular if for each rank one subgroup A/H of G/H containing x, there is an element $a \in x$ such that $o(a) = o(x)$ and $H_A(a) = H_{A/H}(x)$. The subgroup H is a regular subgroup of G if every coset of G/H is regular. A short exact sequence $H \rightarrowtail G \twoheadrightarrow K$ is regular if the image of H is a regular subgroup of G.

We need to verify that this deønition is equivalent to Lyapin's in the event both G and G/H are torsion free. It is easy to see that if a coset is regular in the sense just deøned, then every coset has an element of maximal order. If $x \in G/H$ and both G and G/H are torsion free, then there is a pure rank one subgroup P/H of G/H containing x. Then use the equalities $H_{P/H}(x) = H_{G/H}(x)$ and for $a \in x$, $H_P(a) = H_G(a)$, to get an element $a \in x$ of maximal height. For the converse, we need to apply Lyapin's theorem. If G and G/H are both torsion free and every coset of G/H contains an element of maximal height, let $x \in A/H \subseteq G/H$ with A/H rank one. There is a pure rank one subgroup P/H of G/H with $A/H \subseteq P/H$.

Applying Lyapin's theorem to P/H we get $P = H \oplus R$ for some R, and consequently $A = H \oplus (A \cap R)$. Now pick $a \in x$ such that $a \in A \cap R$ and we get $H_{A/H}(x) = H_A(a)$.

Our aim is to prove that the regular exact sequences are precisely the proper short exact sequences in the relative homological algebra generated by the projective class of all completely decomposable groups. Most of the work lies in proving the following

Lemma. If H is a regular subgroup of G and G/H is a reduced rank one torsion free group, then H is a direct summand of G.

Proof. Let $0 \neq a + H \in G/H$, with $H_G(a) = H_{G/H}(a+H) = \langle k_1, k_2, \ldots\rangle$. Let $F = \{n : k_n \text{ is finite}\}$, $I = \{n : k_n = \infty\}$. For $i \in F$, let $a = p_i^{k_i} b_i$. For $i \in I$, we know $a \in p_i^{\infty} G$, so we can find $b_{i,1} \in p_i^{\infty} G$ with $p_i b_{i,1} = a$. Pick $b_{i,j+1} \in p_i^{\infty} G$ with $p b_{i,j+1} = b_{i,j}$, $j = 1, 2$. Let A be the subgroup of G generated by the sets $\{b_i\}_{i \in F}$ and $\{b_{i,j}\}_{i \in I, j \in Z^+}$. We first show A is torsion free. If $x \in A$, $x = \sum_{i \in J} n_i b_i$ for J some finite set of positive integers, and where for $i \in J \cap I$, $b_i = b_{i,j_i}$. Suppose $mx = 0$ for some $m \neq 0$. Let t_i be a non-negative integer for which $p_i^{t_i} n_i b_i$ is a multple of a. Let $q = \prod_{i \in J} p_i^{t_i}$ and $q_i = q/p_i^{t_i}$. Then we have $0 = qmx = m \sum_{i \in J} q n_i b_i = m \sum_{i \in J} q_i m_i a$, where $p_i^{t_i} n_i b_i = m_i a$. Then $\sum_{i \in J} q_i m_i = 0$ since a has infinite order. We infer that $p_i^{t_i} | m_i$ for $i \in J$. This means $p_i^{t_i} n_i b_i = p_i^{t_i} m_i' a$ for some m_i'. Now $a = p_i^{h_i} b_i$ for some $h_i > 0$. This gives $p_i^{t_i} n_i b_i = p_i^{t_i + h_i} m_i' b_i$. Since b_i has infinite order, we infer that $n_i = p_i^{h_i} m_i'$. Thus $\sum n_i b_i = \sum m_i' a$. This element is either 0 or of infinite order. We conclude that A is torsion free. Now if $h \in H \cap A$, $h = \sum n_i b_i$ as before, and for some integers $m \neq 0, r$ we have $mh = ra$. But $o(a+H) = \infty$ implies $r = 0$. Since A is torsion free, this implies $h = 0$. It remains to show G is generated by the two subgroups A and H. Since G/H has rank one, $G/(A \oplus H)$ is torsion, so it will suffice to show this quotient has no elements of order p for any prime p. Suppose $px \in A \oplus H$. Then $px = \sum n_i b_i + h$, and for some $0 \neq m$, $mpx = ra + mh$. Let m be the smallest such integer. Now if $p | r$, we have $mh \in pH$, and $mh = ph'$ for some $h' \in H$, since H is pure. This means $mx - r'a - h'$ has finite order. But H contains all elements of finite order, since G/H is torsion free. Thus we have $mx \in A \oplus H$, and $mx = r'a + h''$ with $h'' \in H$. Now $p \nmid m$, otherwise $(m/p)px = r'a + h''$ contradicts the choice of m. Thus there are integers s, t with $sp + tm = 1$. Then $x = spx + tmx \in A \oplus H$. We conclude that $G = A \oplus H$, and the lemma is proved. ∎

Theorem 3 Let H be a subgroup of a group G. The short exact sequence $H \rightarrowtail G \twoheadrightarrow G/H$ is proper with respect to the projective class of completely decomposable groups if and only if H is a regular subgroup of G.[2]

[2] Another characterization came to light after publication thanks to a suggestion of E. A. Walker. Theorem. A short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ belongs to $\mathcal{E}(\mathcal{C})$ if and only if for every sequence $\{n_p\}$ of non-negative integers and symbols ∞, indexed by the set of primes, the sequence $\cap_p p^{n_p} A \rightarrowtail \cap_p p^{n_p} B \twoheadrightarrow \cap_p p^{n_p} C$ is exact. By restricting to the relevant set of sequences $\{n_p\}$, this easily generalizes to characterizations of the exact sequences that arise for different classes $\mathcal{C}$ of rank one groups.

Proof. Suppose the sequence is proper. Let $x \in A/H \subseteq G/H$ with A/H rank one. Then $A = H \oplus B$ for some subgroup B of G, and $x = b + H$ for some $b \in B$. Clearly $H_A(b) = H_{A/H}(x)$ and $o(b) = o(x)$. Now suppose the sequence is regular. Let C be any rank one group and $f : C \to G/H$ a homomorphism. Let A/H be the image of f. We will show H is a direct summand of A. Then writing $A = H \oplus B$, $C \to A/H \twoheadrightarrow B \rightarrowtail G \twoheadrightarrow G/H$ factors f through $G \twoheadrightarrow G/H$, showing the sequence is proper. If A/H is torsion free, the lemma says A is a summand of H. Otherwise, $A/H = \sum_p A^p/H$ with $A^p/H \cong Z(p^n)$ for some $0 \leq n \leq \infty$. If $n < \infty$, every coset of A^p/H having a representative of the same order implies H is a summand of A^p. If $n = \infty$, let $a + H \in A^p/H$ have order p. Then we may pick a with $o(a) = p$ and p-height ∞. This implies $a \in D^p \subseteq A^p$ with $D^p \cong Z(p^\infty)$. Since D^p and H have no elements of order p in common we conclude $D^p \cap H = 0$. Then since every proper subgroup of $Z(p^\infty)$ is øfinite, and $(D^p \oplus H)/H$ is inøfinite, we have $D^p \oplus H = A^p$. Thus for every prime p we have a decomposition $A^p = D^p \oplus H$ (with D^p either isomorphic to $Z(p^\infty)$ or cyclic). It follows easily that $A = H \oplus \sum_p D^p$. ■

We do not know the dimension of this relative homological algebra. However, we can determine the dimension of certain groups. We need only consider reduced groups, since all divisible groups are projective.

Theorem 4 If every rank one torsion free subgroup of a reduced group G is free then G has projective dimension 0 or 1 with respect to the projective class of all completely decomposable groups.

Proof. For such a group G, every rank one subgroup is cyclic. Let P be tbe (external) direct sum of all cyclic subgroups of G, and $P \twoheadrightarrow G$ the map induced by inclusions. Clearly $\mathrm{Hom}(C, P) \to \mathrm{Hom}(C, G)$ is an epimorphism for all rank one groups C. Then $K \rightarrowtail P \twoheadrightarrow G$ is a proper projective resolution, since subgroups of direct sums of cyclic groups are again direct sums of cyclic groups. ■

This theorem says, in particular, that all torsion groups have projective dimension ≤ 1. We get a theorem for certain torsion free groups by applying a theorem of Baer [1]. By a homogeneous group we mean a torsion free group all of whose non-zero elements have the same type.

Theorem 5 If G is a homogeneous group then G has projective dimension 0 or 1 with respect to the projective class of all completety decomposable groups.

Proof. Let P be the external direct sum of all pure rank one subgroups of G. Then $K \rightarrowtail P \twoheadrightarrow G$ is a proper exact sequence. Since P is a direct sum of torsion free groups of rank one and of the same type, then every pure subgronp of P is again completely decomposable [1]. Thus K is completely decomposable and $K \rightarrowtail P \twoheadrightarrow G$ is a proper piojective resolution of G. ■

It remains an open question whether a regular subgroup of a completely decomposable group is again completely decomposable.

Since regular sequences are, in particular, pure exact sequences, the injective class contains all algebraically compact groups. It is not known whether the injective class coutains groups which are not algebraically compact.

References

[1] R. Baer, Abelian groups without elements of ønite order. Duke Math. J. 3, 68-122 (1937).

[2] L. Fuchs, Abelian Groups. Budapest 1958.

[3] I. Kaplansky, Projective modules. Ann. of Math. 68, 371-377 (1958).

[4] L. Ya. Kulikov, On direct decompositions of groups [Russian]. Ukrain. Math. Ž. 4, 230-275 and 347-372 (1952)

[5] E. C. Lyapin, On the decomposition of abelian groups into direct sums of rank one groups [Russian] Mat. Sb. 8, 205-237 (1940).

[6] F. Richman C. Walker and E. A. Walker, Projective classes of abelian groups. in: Studies on Abelian Groups, pp.335-343. Paris 1967.

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