Stone Algebra Extensions with Bounded Dense Set*

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Abstract

Stone algebras have been characterized by Chen and Grätzer in terms of triples (B, D, φ), where D is a distributive lattice with 1, B is a Boolean algebra, and φ is a bounded lattice homomorphism from B into the lattice of filters of D. If D is bounded, the construction of these characterizing triples is much simpler, since the homomorphism φ can be replaced by one from B into D itself. The triple construction leads to natural embeddings of a Stone algebra into ones with bounded dense set. These embeddings correspond to a complete sublattice of the distributive lattice of lattice congruences of S. In addition, the largest embedding is a reflector to the subcategory of Stone algebras with bounded dense sets and morphisms preserving the zero of the dense set.

1 Introduction

Stone algebras first gained interest when they were characterized by Grätzer and Schmidt as the solution of a problem of Stone: they are the bounded distributive lattices for which the set of prime ideals satisfies the property that each prime ideal contains a unique minimal prime ideal [9]. These algebras were studied quite extensively in the sixties and early seventies. Recently, Stone algebras with bounded dense sets have arisen in various applications, including conditional event algebras and the study of rough sets and this has led us to investigate this special class of Stone algebras.

One of the main tools for understanding the structure of a Stone algebra was provided by Chen and Grätzer's triple construction [3, 4]. A Stone algebra S is determined as soon as we know its center, its dense set, and how the two sit inside S. Chen and Grätzer showed that this information is carried by a triple (B, D, φ),

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where D is a distributive lattice with 1, B is a Boolean algebra, and φ is a bounded lattice homomorphism from B into the lattice of filters of D. Such a triple gives rise to a Stone algebra whose center is B and whose dense set is D. Conversely, given a Stone algebra with dense set D and center B, there is a natural homomorphism from B into the lattice of filters of D, and this triple yields a Stone algebra isomorphic to the original one. Katriňák and others [10, 11, 12, 13, 14, 19, 20] have simplified this construction and generalized it in various directions. In this paper we use a slight variation of Katriňák's triple construction that permits a more straightforward construction of the corresponding Stone algebra.

If a Stone algebra has a bounded dense set, as the ones that arise in recent applications do, the triple described above can be replaced by a triple (B, D, φ), where D is a bounded distributive lattice, B is a Boolean algebra, and φ is a bounded lattice homomorphism from B *into D itself*, as observed by Katriňák in [10] and studied further by Köhler in [15]. This major simplification of the triple construction led us to the study of embeddings of a Stone algebra into Stone algebras with bounded dense set, which is the primary subject of this paper. We define a *bounded dense extension* of a Stone algebra S to be a Stone algebra monic $\gamma : S \to T$, where the dense set of T has a smallest element and T is generated as a Stone algebra by the image of S together with the smallest element of the dense set of T. We show that the relation defined by $T_1 \leq T_2$ if the map $S \to T_1$ factors through the map $S \to T_2$ orders the set of isomorphism classes of bounded dense extensions of a Stone algebra S. There is a natural order embedding of this poset into the lattice of congruences of S viewed as a bounded lattice. The image of this embedding is a complete and bounded sublattice of the congruence lattice.

The association of a Stone algebra S with its largest bounded dense extension leads to a functor from the category of Stone algebras to the subcategory consisting of Stone algebras with bounded dense set together with those Stone algebra maps that preserve the zero of the dense set which is a reflector. Since this is not a full subcategory, the reflector is not idempotent.

2 Stone Algebras and the Category of Triples

An element x^* in a bounded distributive lattice is the *pseudocomplement* of an element x if $x \wedge y = 0$ exactly when $y \leq x^*$. A *Stone algebra* is a bounded distributive lattice S in which every element has a pseudocomplement and $x^* \vee x^{**} = 1$ for all elements x. An element x of S is *complemented* if $x \vee x^* = 1$. The *center* of S is the largest Boolean sublattice $B(S)$ of S. It consists of the complemented elements of S, or equivalently in the case of a Stone algebra, is the image of S under pseudocomplementation. The *dense set* of S is the set $D(S)$ of elements of S whose pseudocomplement is 0. The dense set of S is a filter (or dual ideal) of S, and in particular, a distributive sublattice with 1. For any distributive lattice L, $\mathcal{F}(L)$ will denote its lattice of filters. If L has a 1, then $\mathcal{F}(L)$ is a bounded lattice.

For a Boolean algebra B and a distributive lattice D with 1, we take a *triple* (B, D, φ) to be a bounded homomorphism $\varphi : B \to \mathcal{F}(D)^\partial$ from the Boolean algebra B to $\mathcal{F}(D)^\partial$, the dual lattice of the filter lattice of D. The dual of a lattice in this sense is the lattice obtained by reversing the order, thus interchanging the roles of the meet and join operations. This slight change in the definition of a triple simplifies the construction of the corresponding Stone algebra. First we get the Stone algebra

$$U = \{(F, b) : F \supseteq \varphi(b)\} \subseteq \mathcal{F}(D)^\partial \times B$$

where U is viewed as a sublattice of $\mathcal{F}(D)^\partial \times B$, and pseudo-complementation is given by $(F, b)^* = (\varphi(b^*), b^*)$. The Stone algebra associated with the triple, is the subalgebra of U given by

$$\mathcal{S}(B, D, \varphi) = \{(\uparrow d \vee \varphi(b), b) : b \in B, d \in D\},$$

where $\uparrow d$ is the principal filter of D generated by d. This Stone algebra has dense set $\{(\uparrow d, 1) : d \in D)\}$ isomorphic to D, and center $\{(\varphi(b), b) : b \in B\}$ isomorphic to B.

Let S be a Stone algebra with dense set D and center B, and let φ be the map $B \to \mathcal{F}(D)$ defined by $\varphi(b) = \uparrow b \cap D$, where $\uparrow b$ is the principal filter of S generated by b. Then φ is an anti-homomorphism, in particular,

$$\varphi(b \vee c) = \uparrow(b \vee c) \cap D = ((\uparrow b) \cap D) \wedge ((\uparrow c) \cap D) = \varphi(b) \wedge \varphi(c),$$

$$\varphi(b \wedge c) = \uparrow(b \wedge c) \cap D = ((\uparrow b) \cap D) \vee ((\uparrow c) \cap D) = \varphi(b) \vee \varphi(c).$$

Now the associated Stone algebra $\mathcal{S}(B, D, \varphi)$ is isomorphic to S by the map

$$\mathcal{S}(B, D, \varphi) \longrightarrow S : (\uparrow d \vee \varphi(b), b) \longmapsto d \wedge b$$

as shown in [3].

The following fundamental property of the lattice of filters is stated in [3], (2.12).

Lemma 1 (Principal Filter Lemma) *If D is a distributive lattice with 1, then for any filter F in the center of the lattice $\mathcal{F}(D)$ and any $d \in D$, the filter $\uparrow d \cap F$ is principal.*

This lemma plays a key role in the study of triples in the following way. If $\varphi : B \to \mathcal{F}(D)^\partial$ is a bounded homomorphism, with B a Boolean algebra, then $\varphi(b)$ is in the center of $\mathcal{F}(D)^\partial$ for all $b \in B$ so in particular, $\uparrow d \cap \varphi(b)$ is principal for all $d \in D$, $b \in B$.

The triple construction gives a one-to-one correspondence between (isomorphism classes) of Stone algebras and triples. This correspondence yields a categorical equivalence with a morphism of triples $f : (B, D, \varphi) \to (C, E, \psi)$ defined to be a pair $f = (g, h)$, where $g : B \to C$ is a Boolean algebra homomorphism and $h : D \to E$ is a distributive lattice with 1 homomorphism, with the property that for each $b \in B$, $\psi(g(b)) \supseteq h(\varphi(b))$. The homomorphism $h : D \to E$ induces the map

$$\mathcal{F}(h) : \mathcal{F}(D)^\partial \to \mathcal{F}(E)^\partial : F \longmapsto \uparrow\{h(d) : d \in F\}.$$

Note that $\mathcal{F}(h)$ is a homomorphism and $\mathcal{F}(h)(\uparrow d) = \uparrow h(d)$. With this notation, the morphism (g, h) gives the diagram

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & \mathcal{F}(D)^{\partial} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(h)} \\ C & \xrightarrow[\psi]{} & \mathcal{F}(E)^{\partial} \end{array}$$

where $\psi(g(b)) \supseteq \mathcal{F}(h)(\varphi(b))$ for all $b \in B$. The condition that (g, h) be a morphism of triples can be stated in any of the following three equivalent ways. The equivalence of *1.* and *2.* below is comment (5.9) in [3].

Lemma 2 *The following are equivalent for a morphism (g, h).*

1. $\mathcal{F}(h)\varphi(b) = \psi g(b) \cap \mathcal{F}(h)(D)$ *for all* $b \in B$.
2. $\mathcal{F}(h)\varphi(b) \subseteq \psi g(b)$ *for all* $b \in B$.
3. $\psi g(b) \cap \uparrow h(d) = \mathcal{F}(h)(\uparrow d \cap \varphi(b))$ *for all* $b \in B$, $d \in D$.

The diagram commutes exactly when $\mathcal{F}(h)(D) = E$, in other words, when $\mathcal{F}(h)$ is a bounded homomorphism. In this situation, we call $f = (g, h)$ a *strong homomorphism* of Stone algebras.

The following theorem of Chen and Grätzer (Theorem 4 in [3]) allows us to characterize subalgebras of Stone algebras in terms of subobjects of the center and the dense set.

Theorem 3 *Let S be a Stone algebra with dense set D and center B. Let E be a sublattice of the lattice D with* 1 *and C a subalgebra of the Boolean algebra B, and call the pair (C, E)* admissible *if $e \vee c \in E$ for all $e \in E, c \in C$. There is a one-to-one correspondence between admissible pairs (C, E) and Stone subalgebras of S. The pair (C, E) corresponds to the subalgebra $\{e \wedge c : e \in E, c \in C\}$, which has dense set E and center C.*

For example, given any sublattice E of D with 1, the pair $(\{0, 1\}, E)$ gives rise to the Stone subalgebra $E \cup \{0\}$ of S. Also, each subalgebra C of B gives rise to the Stone subalgebra $\{x \in S : x^* \in C\}$ of S determined by the pair (C, D).

3 Bounded Dense Extensions of Stone Algebras

Many Stone algebras, including all finite ones, have bounded dense set. A class of Stone algebras with bounded dense set which has recently been the object of some attention arises from Boolean algebras. If B is any Boolean algebra,

$$B^{[2]} = \{(x, y) : x, y \in B, x \leq y\}$$

with component-wise operations is a Stone algebra with dense set

$$D = \{(x, 1) : x \in B\},$$

which is bounded with bottom and top elements $(0, 1)$ and $(1, 1)$, respectively. The Stone algebras $B^{[2]}$ have arisen recently in various applications, including conditional event algebras ([5, 7, 21]), and the study of rough sets ([6, 17, 18]). Stone algebras with bounded dense sets are closed under many constructions, such as direct products and passing from a Stone algebra S to $S^{[2]} = \{(a, b) : a, b \in S, a \leq b\}$.

For the class of Stone algebras with bounded dense set, the triple construction becomes much simpler. The image of the homomorphism φ from $B(S)$ to the lattice $\mathcal{F}(D(S))^{\partial}$ is now entirely contained in the sublattice of principal filters, by the Principal Filter Lemma. Thus the map φ may be considered as a bounded lattice homomorphism from $B(S)$ to $D(S)$ itself. In other words, the natural map $Map(B, D) \rightarrow Map(B, \mathcal{F}(D)^{\partial}) : \alpha \rightarrow \uparrow\alpha$ is a bijection, where $Map(B, D)$ and $Map(B, \mathcal{F}(D)^{\partial})$ are the sets of bounded distributive lattice maps.

Definition 4 *A* bounded triple *is a triple (B, D, φ) where B is a Boolean algebra, D is a bounded distributive lattice, and $\varphi : B \rightarrow D$ is a bounded lattice homomorphism. The Stone algebra arising from a bounded triple (B, D, φ) is*

$$\mathcal{S}_b(B, D, \varphi) = \{(d, b) : d \in D, b \in B, d \leq \varphi(b)\}$$

considered as a sublattice of $D \times B$ with pseudocomplement given by $(d, b)^* = (\varphi(b^*), b^*)$.

Note that the map $b \longmapsto (\varphi(b), b)$ is an isomorphism between B and the center of $\mathcal{S}_b(B, D, \varphi)$, and $D \rightarrow \{(d, 1) : d \in D\} : d \rightarrow (d, 1)$ is an isomorphism between D and the dense set of $\mathcal{S}_b(B, D, \varphi)$. If S is a Stone algebra with center B and dense set D, having a least dense element 0_D, then the inverse of the isomorphism $\mathcal{S}_b(B, D, \varphi) \rightarrow S$ is the map $x \longmapsto (x \vee 0_D, x^{**})$.

Example 5 *Given an arbitrary triple (B, D, φ), then $\varphi : B \rightarrow \mathcal{F}(D)^{\partial}$ is a bounded homomorphism, and thus $(B, \mathcal{F}(D)^{\partial}, \varphi)$ is a bounded triple. This gives rise to the Stone algebra*

$$U = \mathcal{S}_b(B, \mathcal{F}(D)^{\partial}, \varphi) = \{(F, b) : F \supseteq \varphi(b)\} \subseteq \mathcal{F}(D)^{\partial} \times B$$

which naturally contains

$$S = \mathcal{S}(B, D, \varphi) = \{(\uparrow d \vee \varphi(b), b) : b \in B, d \in D\}.$$

Thus any Stone algebra is naturally embedded in a Stone algebra with bounded dense set.

Lemma 6 *Let $S = (B, D, \varphi)$ be a triple. The pair (C, E), where*

$$E = \{(\uparrow d \vee \varphi(b), 1) : d \in D, b \in B\}$$
$$C = \{(\varphi(b), b) : b \in B\}$$

is an admissible pair, and the subalgebra $\overline{S}$ of U determined by this pair is the smallest subalgebra of $U = S_b(B, \mathcal{F}(D)^\partial, \varphi)$ containing S with bounded dense set. Moreover,

$$\overline{S} = \{(\uparrow d \vee \varphi(b_1), b_2) : d \in D, b_1, b_2 \in B, b_1 \leq b_2\}.$$

Proof. It is easy to see that $C \cong B$ is a Boolean algebra and that E is a bounded sublattice of $D(U) = \{(F, 1) : F \in \mathcal{F}(D)\}$. For (C, E) to be an admissible pair in the sense of Theorem 3, $e \vee c$ must belong to E for $e \in E$ and $c \in C$. Thus for $d \in D$ and $b_1, b_2 \in B$ the element $(\uparrow d \vee \varphi(b_1), 1) \vee (\varphi(b_2), b_2)$ must be contained in E. But

$$\begin{aligned}(\uparrow d \vee \varphi(b_1), 1) \vee (\varphi(b_2), b_2) &= (((\uparrow d \vee \varphi(b_1)) \cap \varphi(b_2), 1 \vee b_2) \\ &= (((\uparrow d \cap \varphi(b_2)) \vee \varphi(b_1 \vee b_2), 1).\end{aligned}$$

By the Principal Filter Lemma $\uparrow d \wedge \varphi(b_2)$ is principal, so $(((\uparrow d \wedge \varphi(b_2)) \vee \varphi(b_1 \wedge b_2), 1)$ is an element of E. Also note that the dense set $\{(\uparrow d, 1) : d \in D\}$ of S is contained in E. Thus the subalgebra determined by (C, E) is a Stone algebra with bounded dense set containing S.

To see that (C, E) is the smallest such admissible pair, note that the dense set of any such pair must contain $\{(\uparrow d, 1) : d \in D\}$ and a lower bound of this set. If $(F, 1) \in D(U)$ is a lower bound for $\{(\uparrow d, 1) : d \in D\}$ then $F \supseteq \uparrow d$ for all $d \in D$. This implies $F = D$. Thus the dense set of any such pair must contain all elements of the form (

$$D, 1) \vee (\varphi(b), b) = (D \cap \varphi(b), 1 \vee b) = (\varphi(b), 1).$$

Since E is a sublattice it is closed under meets, so $(\uparrow d, 1) \wedge (\varphi(b), 1) = (\uparrow d \cap \varphi(b), 1)$ is in E.

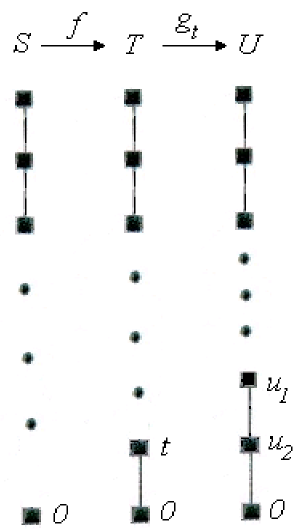
By Theorem 3, the Stone subalgebra $\overline{S}$ of U determined by the pair (C, E) is $\{e \wedge c : e \in E,\ c \in C\}$. Thus

$$\begin{aligned}\overline{S} &= \{(\uparrow d \vee \varphi(a), 1) \wedge (\varphi(b), b) : d \in D,\ a, b \in B\} \\ &= \{(\uparrow d \vee \varphi(a \wedge b), b) : d \in D,\ a, b \in B\} \\ &= \{(\uparrow d \vee \varphi(b_1), b_2) : d \in D,\ b_1, b_2 \in B,\ b_1 \leq b_2\}. \ \Box\end{aligned}$$

Given a Stone algebra S, we have embedded it into the Stone algebra $\overline{S}$ whose dense set is bounded. Note that if D is already bounded, $\overline{S} = S$ since in this case each $\varphi(b)$ is principal and $E = D$.

In the category of Stone algebras, the assignment $S \longmapsto \overline{S}$ cannot be extended to a functor with the property that $\overline{f}$ extends f, since any such extension to morphisms does not preserve composition. To see this, consider the following example.

Example 7 *Let S, T, and U be the chains $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$, $T = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{t, 0\}$, and $U = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{u_1, u_2, 0\}$, as illustrated in the diagram that follows. Let f be the bounded homomorphism from S to T that is the identity on $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Let g_i be the bounded homomorphism from T to U that is the identity on $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and takes t to u_i for $i = 1, 2$. Then $g_1 \circ f = g_2 \circ f$.*



Applying the operator to S gives the lattice $\overline{S} \cong T$. If the assignment were extended to the morphisms, we would have $\overline{(g_1 \circ f)} = \overline{(g_2 \circ f)}$. On the other hand, the only possibility for $\overline{f} : \overline{S} \to \overline{T} \cong T$ extending f is the isomorphism of $\overline{S}$ and T. Since both T and U are already bounded, we also have that $\overline{g_i} = g_i$, $i = 1, 2$. So we have

$$\overline{S} \cong T \overset{g_1}{\to} U \text{ and } \overline{S} \cong T \overset{g_2}{\to} U$$

and $\overline{g_1} \circ \overline{f} = \overline{g_2} \circ \overline{f}$ if and only if $g_1 = g_2$, which is clearly not the case.

Given a Stone algebra $S = \mathcal{S}(B, D, \varphi)$, $\overline{S}$ was defined to be the smallest subalgebra of $F(D)^\partial \times B$ that has bounded dense set and contains S. We may ask whether $\overline{S}$ is, in some sense, the smallest Stone algebra with bounded dense set containing S. Before we can answer this question we need to explore a larger class of extensions, the bounded dense extensions of a Stone algebra S.

Definition 8 *A* bounded dense extension *of a Stone algebra S is a Stone algebra T and a Stone algebra monic $\gamma : S \to T$ satisfying*

1. *The dense set E of T has a smallest element.*
2. *The algebra T is generated as a Stone algebra by $\gamma(S) \cup \{0_E\}$.*

Theorem 9 *Let $\gamma : S \to T$ be a Stone algebra monomorphism and suppose $E = D(T)$ is bounded. Let $B = B(S)$, $D = D(S)$ and $C = B(T)$. Then $\gamma : S \to T$ is a bounded dense extension if and only if*

1. *The restriction $\gamma|B$ is an isomorphism of Boolean algebras $B \cong C$, and*
2. *$E = \{\gamma(d) \wedge (c \vee 0_E) : d \in D, c \in C\}$.*

Proof. Assume $\gamma|B$ is an isomorphism of Boolean algebras $B \cong C$ and $E = \{\gamma(d) \wedge (c \vee 0_E) : d \in D, c \in C\}$. We are given that γ is a Stone algebra monic and that the dense set E of T has a smallest element. But $T = \{e \wedge c : e \in E, c \in C\}$ so we have

$$T = \{(\gamma(d) \wedge (\gamma(b_1) \vee 0_E)) \wedge \gamma(b_2) : d \in D, b_1, b_2 \in B\}$$

is generated as a Stone algebra by $\gamma(S)$ and 0_E.

Now assume $\gamma : S \to T$ is a bounded dense extension, so that T is generated as a Stone algebra by $\gamma(S)$ and 0_E. Let $A = \gamma(B)$ and

$$F = \{\gamma(d) \wedge (\gamma(b) \vee 0_E) : d \in D, \ \ b \in B\}.$$

Then $A \subseteq C$. It is easy to show that (A, F) is an admissible pair as defined in Theorem 3, and that both $\gamma(S)$ and 0_E are contained in the subalgebra of T generated by the pair (A, F). It then follows from Theorem 3 and the definition of bounded dense extension that

$$\begin{aligned} C &= A = \gamma(B) \cong B \text{ and} \\ E &= F = \{\gamma(d) \wedge (c \vee 0_E) : d \in D, c \in C\} \end{aligned}$$

as desired. □

Example 10 *Let $S = \mathcal{S}(B, D, \varphi)$. Then $\overline{S} = \{(\uparrow d \vee \varphi(b_1), b_2) : d \in D, \ b_1, b_2 \in B, \ b_1 \leq b_2\}$ with the map $\overline{\gamma} : S \to \overline{S} : (\uparrow d \vee \varphi(b), b) \longmapsto (\uparrow d \vee \varphi(b), b)$ is a bounded dense extension. Certainly $\overline{\gamma}$ is a monomorphism which restricts to an isomorphism between the centers. The dense set $E = \{(\uparrow d \vee \varphi(b), 1) : d \in D, b \in B\}$ of $\overline{S}$ is bounded, having smallest element $(\varphi(0), 1) = (D, 1)$. Also*

$$\begin{aligned} (\uparrow d \vee \varphi(b), 1) &= (\uparrow d, 1) \wedge (\varphi(b) \cap D, b \vee 1) \\ &= \overline{\gamma}((\uparrow d, 1)) \wedge ((\varphi(b), b) \vee (D, 1)) \end{aligned}$$

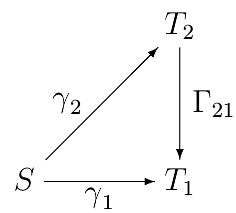
Example 11 *Let $S = \mathcal{S}(B, D, \varphi)$. Then $\overline{\overline{S}} = \mathcal{S}_b(B, S, i)$ with the map*

$$\overline{\overline{\gamma}} : S \to \overline{\overline{S}} : (\uparrow d \vee \varphi(b), b) \longmapsto ((\uparrow d \vee \varphi(b), b), b)$$

is a bounded dense extension. Clearly $\overline{\overline{\gamma}}$ *is a monomorphism which restricts to an isomorphism between the centers and* $D\left(\overline{\overline{S}}\right) \cong S$ *is bounded with smallest element* $((D,0),1)$*. An arbitrary element of* $D\left(\overline{\overline{S}}\right)$ *is of the form*

$$((\uparrow d \vee \varphi(b), b), 1) = \overline{\overline{\gamma}}((\uparrow d, 1)) \wedge (((\varphi(b), b), b) \vee ((D, 0), 1)).$$

Definition 12 *Let* $\gamma_i : S \to T_i$ *be bounded dense extensions for* $i = 1, 2$*. Then* $T_1 \leq T_2$ *if there exists a strong homomorphism of Stone algebras* $\Gamma_{21} : T_2 \to T_1$ *such that* $\Gamma_{21} \circ \gamma_2 = \gamma_1$*, i.e. the diagram*



(1)

commutes and $\Gamma_{21}(0_{D(T_2)}) = 0_{D(T_1)}$*.*

We identify the two extensions when Γ_{21} is an isomorphism.

Theorem 13 *Given a Stone algebra* S*, the set of all bounded dense extensions of* S *is partially ordered by the relation* $\leq$*.*

Proof. We may assume the center of T_i is B and that $\gamma_i|B$ is the identity. Let $E_i = D(T_i)$. To see that the relation is an order, note that for $b \in B$, $\gamma_1(b) = \gamma_2(b) = b$ so $\Gamma_{21}|B$ is the identity on B. It is immediately clear that $\leq$ is reflexive and transitive. To prove antisymmetry we show that Γ_{21} is unique whenever it exists. Since every element of E_i is of the form $\gamma_i(d) \wedge (b \vee 0_{E_i})$ and since Γ_{21} is a strong homomorphism, $\Gamma_{21}|E_2 : E_2 \to E_1$ by

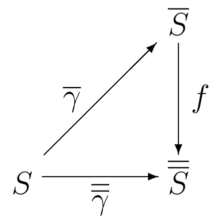
$$\begin{aligned} \Gamma_{21}(\gamma_2(d) \wedge (b \vee 0_{E_2})) &= \Gamma_{21}\gamma_2(d) \wedge (\Gamma_{21}(b) \vee \Gamma_{21}(0_{E_2})) \\ &= \gamma_1(d) \wedge (b \vee 0_{E_1}). \end{aligned}$$

This completely determines Γ_{21}. Thus for any two bounded dense extensions there is at most one strong homomorphism Γ_{21} from T_2 to T_1.

Now suppose $T_1 \leq T_2$ and $T_2 \leq T_1$. Then $\Gamma_{21} \circ \Gamma_{12}$ shows that $T_2 \leq T_2$. But so does the identity map, so by uniqueness $\Gamma_{21} \circ \Gamma_{12}$ is the identity on T_2. Similarly $\Gamma_{12} \circ \Gamma_{21}$ is the identity on T_1. Therefore Γ_{21} is an isomorphism and the extensions are considered equal. □

In the next section we will show that $\overline{S}$ is the smallest bounded dense extension of S in this partial order. However, the following theorem enables us to show that $\overline{S}$ is not always a subalgebra of other bounded dense extensions of S.

Theorem 14 *Let $S = \mathcal{S}(B, C, \varphi)$ be a Stone algebra, $\overline{S} = \mathcal{S}_b(B, E, \overline{\varphi})$ the smallest subalgebra of $\mathcal{S}(B, \mathcal{F}(D)^\partial, \varphi)$ containing S, and $\overline{\overline{S}} = \mathcal{S}_b(B, S, i)$ where $i : B \to S$ is inclusion. Then there is an embedding $f : \overline{S} \to \overline{\overline{S}}$ such that the diagram*



(2)

commutes if and only if the ideal

$$A = \{b \in B : \varphi(b^*) \text{ is principal}\}$$

of B is a principal ideal.

Proof. Suppose $f : \overline{S} \to \overline{\overline{S}}$ satisfies (2). Then

$$\begin{array}{rcl} f((\varphi(b), b)) & = & f(\overline{\gamma}(\varphi(b), b)) \\ & = & \overline{\overline{\gamma}}((\varphi(b), b)) \\ & = & ((\varphi(b), b), b) \end{array}$$

and

$$\begin{array}{rcl} f((\uparrow d, 1)) & = & f(\overline{\gamma}((\uparrow d, 1))) \\ & = & \overline{\overline{\gamma}}((\uparrow d, 1)) \\ & = & ((\uparrow d, 1), 1) \end{array}$$

Now $f((D, 1)) = (s_0, 1)$ for some $s_0 = (\varphi(b_0) \vee \uparrow d_0, b_0) \in S$ and $(D, 1) \le (\uparrow d, 1)$ for all $d \in D$ so

$$f((D, 1)) = (s_0, 1) \le ((\uparrow d, 1), 1)$$

for all $d \in D$. It follows that $\varphi(b_0) \vee \uparrow d_0 = D$ and thus $\varphi(b_0^*) = \varphi(b_0)^* \subseteq \uparrow d_0$, implying by the Principal Filter Lemma that $\varphi(b_0^*)$ is principal and hence that $b_0 \in A$.

Now let $b \in A$. We want to show that $b \le b_0$. Now $b \in A$ implies that $\varphi(b) \vee \uparrow d = D$ for some $d \in D$. Then

$$\begin{array}{rcl} f((D, 1)) & = & f((\varphi(b) \vee \uparrow d, 1)) \\ & = & f((D, 1) \vee ((\varphi(b), b) \wedge (\uparrow d, 1))) \\ & = & ((D, b_0), 1) \vee (((\varphi(b), b), b) \wedge ((\uparrow d, 1), 1))) \\ & = & ((D, b_0 \vee b), 1). \end{array}$$

So $b_0 \vee b = b_0$ and $b \le b_0$. On the other hand, if $b \le b_0$ then $b^* \ge b_0^*$ so $\varphi(b^*) \subseteq \varphi(b_0^*)$ and again by he Principal Filter Lemma, $\varphi(b^*)$ is principal and $b \in A$. *We* have shown that $A = \downarrow b_0$.

For the converse, let $b_0 \in B$ be the principal generator of the ideal A and define $f : \overline{S} \to \overline{\overline{S}}$ by the diagram

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & E \\ {\scriptstyle id_B}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[i]{} & S \end{array}$$

where

$$f(\uparrow d \vee \varphi(b)) = (\uparrow d \vee \varphi(b), b \vee b_0).$$

First we show that f is well defined. Note that an alternate description of A is $A = \{b \in B : \uparrow d \vee \varphi(b) = D \text{ for some } d \in D\}$. Suppose $\uparrow d_1 \vee \varphi(b_1) = \uparrow d_2 \vee \varphi(b_2)$. Then $D = \uparrow d_2 \vee \varphi(b_2 \wedge b_1^*)$ and thus $b_0 \geq b_2 \wedge b_1^*$. Similarly, $b_0 \geq b_1 \wedge b_2^*$. Since b_0 contains the symmetric difference of b_1 and b_2,

$$b_0 \vee b_1 = b_0 \vee b_2 = b_0 \vee (b_1 \wedge b_2).$$

Thus f is well defined.

Now in order to show that (id_B, f) is a bounded triple morphism, we need that $f\varphi(b) \geq i(b)$. But $f(\varphi(b)) = (\varphi(b), b \vee b_0) \geq (\varphi(b), b) = i(b)$.

Finally, we check that (2) commutes.

$$\begin{aligned} f(\overline{\gamma}(\uparrow d \vee \varphi(b), b)) &= f((\uparrow d \vee \varphi(b), b)) \\ &= f((\uparrow d, 1) \wedge (\varphi(b), b)) \\ &= f((\uparrow d, 1)) \wedge (i(id_B(b)), id_B(b)) \\ &= ((\uparrow d, 1), 1) \wedge ((\varphi(b), b), b) \\ &= ((\varphi(b) \vee \uparrow d, b), b) \\ &= \overline{\overline{\gamma}}(\uparrow d \vee \varphi(b), b)). \ \square \end{aligned}$$

Example 15 *Take B to be the Boolean algebra of all finite and cofinite subsets of an infinite set X, take D to be the chain $\mathbb{N}^{\mathrm{op}}$, and define $\varphi : B \to D$ by $\varphi(x) = D$ if x is finite, and $\varphi(x) = \{1\}$ if x is infinite. Clearly $\varphi(b^*)$ is principal if and only if b is finite, so the ideal A of the theorem is a non-principal ideal. Thus for $S = \mathcal{S}(B, D, \varphi)$ we have that $\overline{S}$ is not a subalgebra of $\overline{\overline{S}}$.*

4 The Lattice of Bounded Dense Extensions

In the partial order on bounded dense extensions of S, we have shown that the one determined by the bounded triple (B, S, i), where i is the inclusion of the center B in

S, is the largest bounded dense extension of S. In this section we show that the set of all bounded dense extensions of a Stone algebra S is a complete bounded distributive lattice by identifying it with a sublattice of a lattice of congruences and we observe that $\overline{S}$ is the smallest bounded dense extension of S.

Theorem 16 *Let S be a Stone algebra with center B. For each lattice congruence θ of S such that the natural map $D(S) \to S/\theta$ is one-to-one, there is a bounded dense extension $\gamma_\theta : S \to S_b(B, S/\theta, \eta_\theta)$ where η_θ is the natural map $B \to S/\theta : b \longmapsto [b]_\theta$ and $\gamma_\theta(s) = ([s]_\theta\,, s^{**})$ for $s \in S$. Moreover, every bounded dense extension can be obtained in this way.*

Proof. First recall that

$$\mathcal{S}(B, S/\theta, \eta_\theta) = \{(e, b) : e \in S/\theta, b \in B, e \leq \eta_\theta(b)\}$$

and note that $[s]_\theta \leq [s^{**}]_\theta$ so γ_θ is well defined. Now for $b \in B$, $\gamma(b) = ([b]_\theta, b^{**}) = ([b]_\theta, b)$ and it is clear that γ_θ is one-to-one on B. By hypothesis, γ_θ is one-to-one on D. It follows that γ_θ is one-to-one.

For $s \in S$, $s = b \wedge d$ for some $b \in B$, $d \in D$. As a sublattice of $\mathcal{S}(B, S/\theta, \eta_\theta)$, $S/\theta = \{([s]_\theta, 1) : s \in S\}$. Then

$$\begin{aligned}
\gamma_\theta(d) \wedge (b \vee 0_E) &= \{([d]_\theta, 1) \wedge (([b]_\theta, b) \vee ([0]_\theta, 1)) \\
&= ([d]_\theta, 1) \wedge ([b]_\theta, 1) \\
&= ([d \wedge b]_\theta, 1) = ([s]_\theta, 1).
\end{aligned}$$

It follows that

$$D\left(\mathcal{S}(B, S/\theta, \eta_\theta)\right) = S/\theta = \{\gamma_\theta(d) \wedge (b \vee 0_E) : d \in D, b \in B\}$$

and that $\mathcal{S}(B, S/\theta, \eta_\theta)$ is a bounded dense extension of S.

For the converse, let $\gamma : S \to T = \mathcal{S}(B, E, \psi)$ be a bounded dense extension and define θ to be the congruence $\{(x, y) : \gamma(x) \vee 0_E = \gamma(y) \vee 0_E\}$. Then we have the diagram

$$\begin{array}{ccc}
B & \xrightarrow{\ \eta\ } & S/\theta \\
\gamma \downarrow & & \downarrow \sigma \\
\gamma(B) & \xrightarrow[\ \psi\ ]{} & E
\end{array}$$

where $\sigma([x]_\theta) = \gamma(x) \vee 0_E$. It is easy to see that σ is a well-defined bounded homomorphism and that the diagram commutes. Also $\gamma|B$ is an isomorphism. We need σ an isomorphism as well. Now $E = \{\gamma(d) \wedge (\gamma(b) \vee 0_E) : b \in B, d \in D\}$ and

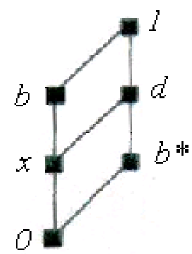
$$\begin{aligned}
\gamma(d) \wedge (\gamma(b) \vee 0_E) &= (\gamma(d) \vee 0_E) \wedge (\gamma(b) \vee 0_E) \\
&= \gamma(d \wedge b) \vee 0_E = \sigma([d \wedge b]_\theta).
\end{aligned}$$

Thus σ is onto. Clearly σ is one-to-one and $T \cong \mathcal{S}(B, S/\theta, \eta_\theta)$. Also it is clear that γ_θ followed by this isomorphism is γ. $\square$

Since the congruences that arise in the previous theorem are determined largely by their values on B the question arises whether or not the lattice of bounded dense extensions of S can be recognized within the lattice of congruences on B. The following example shows this is not the case.

Example 17 *First we note that if θ is a lattice congruence on any Stone algebra satisfying $\theta|D = \triangle_D$, then $\theta|D \subseteq \ker\varphi$ where $\varphi : B \to \mathcal{F}(D)^\partial : b \mapsto \uparrow b \cap D$. To see this, let $b, c \in B$. Then $b\theta c$ implies that $d \vee b\theta d \vee c$ for all $d \in D$. Thus $d \vee b = d \vee c$ for all $d \in D$ from which it follows that $\uparrow b \cap D = \uparrow c \cap D$. Thus $\varphi(b) = \varphi(c)$, or $b\,(\ker\varphi)\,c$.*

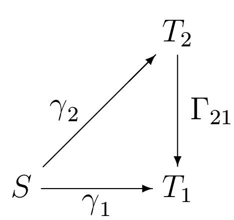
Now let $S = \mathbf{2} \times \mathbf{3}$.



The relations θ_1 corresponding to the partition $\{1, b\}, \{x, d\}, \{0, b^*\}$ and θ_2 corresponding to the partition $\{1, b\}, \{x, d, 0, b^*\}$ are both lattice congruences on S. Also $\theta_i|D = \triangle_D$ and $\theta_i|B = \ker\varphi$ for $i = 1, 2$ where $\varphi : B \to \mathcal{F}(D)^\partial : b \mapsto \uparrow b \cap D$. This illustrates that it is impossible to recognize the bounded dense extensions of S within $Con(B)$.

The following lemma shows that a map between bounded dense extensions is determined by maps on the dense sets together with the triple maps for the bounded dense extensions. Thus some of the advantage of the simpler maps used for bounded triples is available for bounded dense extensions of arbitrary Stone algebras.

Lemma 18 *The commutative diagram*



(3)

where T_1 and T_2 are bounded dense extensions of S and Γ_{21} is a strong homomorphism, *is equivalent to the commutative diagram*

$$\begin{array}{ccccc} & & E_2 & & \\ & \overset{\psi_2}{\nearrow} & \big\downarrow \Gamma_{21} & \overset{\gamma_2}{\nwarrow} & \\ B & \xrightarrow[\psi_1]{} & E_1 & \xleftarrow[\gamma_1]{} & D \end{array} \tag{4}$$

where $B = B(S)$, $D = D(S)$, ψ_1, ψ_2 *and* Γ_{21} *are strong homomorphisms,* γ_1 *and* γ_2 *are monic,* $E_i = D(T_i)$ *and* E_i *is generated by* $\psi_i(B) \cup \gamma_i(D) \cup \{0_{E_i}\}$, $i = 1, 2$ *and* $\psi_2(b) \vee \gamma_2(d) \in \gamma_2(D)$ *for all* $b \in B$, $d \in D$.

Proof. Given (3) we may assume that the center of each T_i is B. Denote the dense set of T_i by E_i and the bounded triple map for T_i by ψ_i for $i = 1, 2$. It is clear that the left triangle of (4) exactly describes the map Γ_{21} in (3), including the property $\Gamma_{21}(0_{E_2}) = 0_{E_1}$. Also the right triangle of (4) is just the restriction of (3) to the dense sets. Finally, the property $\psi_i(b) \vee \gamma_i(d) \in \gamma_i(D)$ says that $(\gamma_i(B), \gamma_i(D))$ is an admissible pair for T_i, $i = 1, 2$.

Given (4), (B, E_i, ψ_i) are bounded triples and thus correspond to Stone algebras $T_i = \mathcal{S}_b(B, E_i, \psi_i)$ with bounded dense sets. Also Γ_{21} induces a map $\Gamma_{21} : T_2 \to T_1$ corresponding to

$$\begin{array}{ccc} B & \xrightarrow{\psi_2} & E_2 \\ {\scriptstyle id_B}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma_{21}} \\ B & \xrightarrow[\psi_1]{} & E_1 \end{array}$$

Clearly Γ_{21} is an isomorphism between the centers; it is a strong homomorphism since $\Gamma_{21} : E_2 \to E_1$ is bounded. Also $\psi_i(b) \vee \gamma_i(d) \in \gamma_i(D)$ for all $b \in B, d \in D$, implies that $(B, \gamma_i(D))$ is an admissible pair in T_i. The fact that E_i is generated by $\psi_i(B) \cup \gamma_i(D) \cup \{0_{E_i}\}$ now implies that $\gamma_i : S_i = S(B, \gamma_i(D)) \to T_i$ are bounded dense extensions. Finally, since γ_1 and γ_2 are monic and the right triangle of (4) commutes, it follows that Γ_{21} is an isomorphism from $\gamma_2(D)$ to $\gamma_1(D)$. We obtain the diagram

$$\begin{array}{ccc} B & \xrightarrow{\varphi_2} & \mathcal{F}(\gamma_2(D))^{\partial} \\ {\scriptstyle id_B}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{F}(\Gamma_{21})} \\ B & \xrightarrow[\varphi_1]{} & \mathcal{F}(\gamma_1(D))^{\partial} \end{array}$$

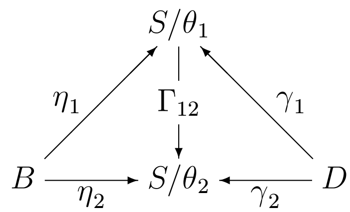
where $\varphi_i(b) = \{\gamma_i(d) : \psi_i(b) \le \gamma_i(d)\}$. This diagram is a triple map if for all $b \in B$, $\mathcal{F}(\Gamma_{21})\varphi_2(b) \subseteq \varphi_1(b)$. Now

$$\mathcal{F}(\Gamma_{21})\varphi_2(b) = \uparrow\{\Gamma_{21}(\gamma_2(d)) : \psi_2(b) \le \gamma_2(d)\}.$$

Suppose $\gamma_2(d) \geq \psi_2(b)$, then $\Gamma_{21}(\gamma_2(d)) \geq \Gamma_{21}(\psi_2(b))$ and by commutativity of the two triangles in (4), $\gamma_1(d) \geq \psi_1(b)$, that is $\Gamma_{21}(\gamma_2(d)) = \gamma_1(d) \in \varphi_1(b)$, and we have the desired inclusion. It follows that $S_2 \cong S_1$ and Γ_{21} carries T_2 to T_1 so that we have (3). □

Theorem 19 *The poset of bounded dense extensions of* $S = \mathcal{S}(B, D, \varphi)$ *is a bounded distributive lattice with* $1 = \mathcal{S}(B, S, i)$ *and* $0 = \mathcal{S}(B, S/\theta_D, \eta)$*, where* θ_D *is the congruence defined by* $x\theta_D y$ *if and only if* $\uparrow x \cap D = \uparrow y \cap D$.

Proof. Let $\mathcal{E}(S)$ denote the poset of bounded dense extensions of S and let $\mathcal{C}(S)$ denote the set of all lattice congruences of S, so that $D(S) \to S/\theta$ is one-to-one. Then by Theorem 16, we have a bijection $\mathcal{C}(S) \to \mathcal{E}(S) : \theta \longmapsto \gamma_\theta$ where $\gamma_\theta : S \to \mathcal{S}_b(B, S/\theta, \eta)$. If $\theta_1 \leq \theta_2$ are congruences in $C(S)$ then we get the diagram



(5)

where η_i and γ_i are the inclusion maps into S followed by the quotient map $S \to S/\theta_i$ for $i = 1, 2$ and Γ_{12} is the map $S/\theta_1 \to S/\theta_2$ allowed by $\theta_1 \leq \theta_2$. Consequently the diagram is commutative. It also follows easily from the given properties that the left triangle consists of bounded maps and γ_1 and γ_2 are monic. [Check the rest of the lemma conditions that don't follow directly from the proof of Theorem 16.] Thus we have that $S_b(B, S/\theta_1, \eta_1) \geq S(B, S/\theta_2, \eta_2)$ and $\mathcal{E}(S) \cong \mathcal{C}(S)^{op}$ as a poset.

Now, the poset $\mathcal{C}(S)$ sits in the complete distributive lattice $\mathbf{Con}(S)$ of all lattice congruences on S. In fact, $\mathcal{C}(S)$ is a complete lattice ideal of $\mathbf{Con}(S)$. To see this, it is clear that the trivial congruence on S, Δ_S, belongs to $\mathcal{C}(S)$. Also, if $\theta_1 \leq \theta_2$ and $D(S) \to S/\theta_2$ is one-to-one, then $D(S) \to S/\theta_1$ must also be one-to-one. Thus $\theta_2 \in \mathcal{C}(S)$ implies $\theta_1 \in \mathcal{C}(S)$. Finally, if $\mathcal{C}$ is any subcollection of $\mathcal{C}(S)$ then it is easy to see that $D(S) \to S/\left(\bigvee \mathcal{C}\right)$ must be one-to-one since $\bigvee \mathcal{C}$ is the transitive closure of the congruences in $\mathcal{C}$.

It now follows that $\mathcal{C}(S)$ is a complete distributive lattice and thus that $\mathcal{E}(S) \cong \mathcal{C}(S)^{op}$ is also a complete distributive lattice. Since Δ_S is the smallest element of $\mathcal{C}(S)$, $\mathcal{S}_b(B, S/\Delta_S, \eta_\Delta) = \mathcal{S}_b(B, S, i)$ where $i : B \hookrightarrow S$ is the inclusion, is the largest element of $\mathcal{E}(S)$. It is easy that the filter congruence θ_D on S generated by D, that is $x\theta_D y$ if and only if $\uparrow x \cap D = \uparrow y \cap D$, is the largest element in $\mathcal{C}(S)$ and thus the corresponding element of $\mathcal{E}(S)$ is the 0 of the lattice $\mathcal{E}(S)$. □

Corollary 20 $\overline{S}$ *is the smallest element of* $\mathcal{E}(S)$.

Proof. The congruence corresponding to $\gamma : S \to \overline{S} : s = (s \vee s^*) \wedge s^{**} \longmapsto (\uparrow(s \vee s^*) \vee \varphi(s^{**}), s^{**})$ is $s \sim t$ if and only if

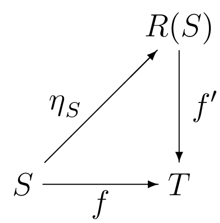
$$\begin{aligned}
(\uparrow(s \vee s^*) \vee \varphi(s^{**}), s^{**}) \vee (D, 1) &= (\uparrow(t \vee t^*) \vee \varphi(t^{**}), t^{**}) \vee (D, 1) \\
(\uparrow(s \vee s^*) \vee (\uparrow s^{**} \cap D) &= (\uparrow(t \vee t^*) \vee (\uparrow t^{**} \cap D) \\
(\uparrow(s \vee s^*) \vee \uparrow s^{**}) \cap D &= (\uparrow(t \vee t^*) \vee \uparrow t^{**}) \cap D \\
\uparrow(s \vee s^*) \wedge s^{**}) \cap D &= \uparrow(t \vee t^*) \wedge t^{**}) \cap D \\
\uparrow s \cap D &= \uparrow t \cap D
\end{aligned}$$

This is the condition for the congruence $s\theta_D t$. □

5 The Category of Stone Algebras with Bounded Dense Set

It is now easy to show that $S \mapsto \overline{\overline{S}}$ naturally induces a reflective functor from the category of Stone algebras and Stone algebra homomorphisms to the subcategory of Stone algebras with bounded dense set and strong homomorphisms.

Definition 21 *A subcategory $\mathcal{S}'$ of a category $\mathcal{S}$ is* reflective *if there is a functor $R : \mathcal{S} \to \mathcal{S}'$, called a* reflector, *and a natural transformation $\eta : I_{\mathcal{S}} \to R$ from the identity functor of $\mathcal{S}$ to the functor R, with the property that for any map $f : S \to T$ in the category $\mathcal{S}$ with T an object of $\mathcal{S}'$, there exists a unique map $f' : R(S) \to T$ in $\mathcal{S}'$ such that $f' \circ \eta_S = f$, that is, the diagram*



commutes.

Theorem 22 *The category $\mathcal{S}_b$ of Stone algebras with bounded dense set and strong homomorphisms is a reflective subcategory of the category $\mathcal{S}$ of Stone algebras with Stone algebra homomorphisms, with reflector $R : \mathcal{S} \to \mathcal{S}_b$ given by*

$$R(S) = \overline{\overline{S}} = \mathcal{S}_b(B, S, i) \text{ and } R(f) = \overline{\overline{f}} = (f, f),$$

together with the natural transformation

$$\eta_S = \overline{\overline{\gamma}} : S \to \overline{\overline{S}}.$$

Proof. It suffices to show that for every Stone lattice homomorphism $f : S \to T$, with T having a bounded dense set, there exists a unique strong homomorphism $f' : \overline{\overline{S}} \to T$ with commuting diagram

$$\begin{array}{ccc} & & \overline{\overline{S}} \\ & \overset{\overline{\overline{\gamma_S}}}{\nearrow} & \Big\downarrow{\scriptstyle f'} \\ S & \xrightarrow[f]{} & T \end{array} \tag{6}$$

that is, $f' \circ \overline{\overline{\gamma_S}} = f$ ([1], I.18, Theorem 2). In the case f is one-to-one this follows immediately from Theorem 13, since the image of f is contained in a bounded dense extension of S. In the general case, notice first that if f' exists it is unique since $\overline{\overline{S}}$ is generated by $\overline{\overline{\gamma_S}}(S)$ together with the zero of the dense set of $\overline{\overline{S}}$.

Given $f : S \to T$ any Stone map between arbitrary Stone algebras, we define $\overline{\overline{f}} = (f, f) : \overline{\overline{S}} \to \overline{\overline{T}}$ by the bounded triple map

$$\begin{array}{ccc} B & \xrightarrow{i} & S \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ C & \xrightarrow[i]{} & T \end{array}$$

It is easy to check that $\overline{\overline{f}}$ is a strong homomorphism and that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\overline{\overline{\gamma_S}}} & \overline{\overline{S}} \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle \overline{\overline{f}}} \\ T & \xrightarrow[\overline{\overline{\gamma_T}}]{} & \overline{\overline{T}} \end{array}$$

commutes. Now in the case that T has bounded dense set, T is a bounded dense extension of itself and by Theorem 13 there is a strong map $\Gamma : \overline{\overline{T}} \to T$ such that the triangle

$$\begin{array}{ccc} & & \overline{\overline{T}} \\ & \overset{\overline{\overline{\gamma_T}}}{\nearrow} & \Big\downarrow{\scriptstyle \Gamma} \\ T & \xrightarrow[1]{} & T \end{array}$$

commutes. Then $f' = \Gamma \circ \overline{\overline{f}}$ is the desired strong homomorphism solving diagram 6.

6 Some Examples

1. Let D be any distributive lattice without 0 and with 1. Let B be the two element Boolean algebra and φ the only bounded morphism from B to $\mathcal{F}(D)^\partial$. Then $S = \mathcal{S}(B, D, \varphi)$ is just D with a bottom added and $\overline{S}$ is S with a bottom adjoined to its dense set.

2. Let S be a Stone algebra, D a distributive lattice with 1, and $\varphi : S \to \mathcal{F}(D)^\partial$ a bounded homomorphism whose range is contained in the sublattice

$$(D : D) = \{F \in \mathcal{F}(D) : F \cap \uparrow d \text{ is principal for all } d \in D\}$$

of $\mathcal{F}(D)^\partial$. Then $\mathcal{S}(S, D, \varphi) = \{(\uparrow d \vee \varphi(s), s) : d \in D, s \in S\}$ is a sublattice of the product $\mathcal{F}(D)^\partial \times S$ with coordinatewise operations, which becomes a Stone algebra with $(\uparrow d \vee \varphi(s), s)^* = (\varphi(s^*), s^*)$. Notice that $(D : D)$ contains the center of $\mathcal{F}(D)^\partial$ as well as the principal filters of D.

Given a Stone algebra $S = \mathcal{S}(B, D, \varphi)$,

$$\begin{aligned}
\overline{\overline{S}} &= \mathcal{S}(B, S, i) = \{(s, b) : s \leq b, s \in S, b \in B\} \\
&\subseteq S \times B = \{(\uparrow d \vee \varphi(b_1), b_1, b_2) : d \in D, b_i \in B, b_1 \leq b_2\} \\
&\subseteq \mathcal{F}(D)^\partial \times B \times B.
\end{aligned}$$

Now if we define

$$\overline{\overline{\varphi}} : B^{[2]} \to \mathcal{F}(D)^\partial : (b_1, b_2) \mapsto \varphi(b_1)$$

then $\overline{\overline{\varphi}}$ is a bounded homomorphism of the Stone algebra $B^{[2]} = \overline{\overline{B}}$ into $\mathcal{F}(D)^\partial$ whose image is contained in the center of $\mathcal{F}(D)^\partial$. We get $\mathcal{S}(B^{[2]}, D, \overline{\overline{\varphi}}) = \{(\uparrow d \vee \varphi(b_1), b_1, b_2) : b_1 \leq b_2\} = \overline{\overline{S}}$. In other words,

$$\mathcal{S}(\overline{\overline{B}}, D, \overline{\overline{\varphi}}) = \overline{\overline{\mathcal{S}(B, D, \varphi)}}.$$

In fact, for any bounded homomorphism $\varphi : S \to \mathcal{F}(D)^\partial$ into $(D : D)$, we get the Stone algebra $S_0 = \mathcal{S}(S, D, \varphi)$, which then gives rise to the Stone algebra $S_1 = \mathcal{S}(\overline{\overline{S}}, D, \overline{\overline{\varphi}})$, where $\overline{\overline{\varphi}} : \overline{\overline{S}} \to \mathcal{F}(D)^\partial$ is given by $\overline{\overline{\varphi}}(s, b) = \varphi(s)$. Iterating this process we get an increasing chain of structures

$$S_0 \subset S_1 \subset S_2 \subset \cdots .$$

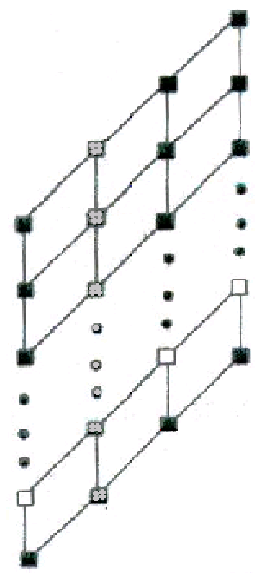
In the case where $S_0 = \mathcal{S}(B, D, \varphi)$ is a Stone algebra obtained using the original triple construction, the sequence obtained is the same as the one obtained by iterating the application of $\overline{\overline{(\ )}}$ to S_0. Finally, in the case where $S_0 = B$ is a Boolean algebra, the sequence obtained is

$$B = B_1 \subset B_2 \subset B_3 \subset \cdots \subset B_n \subset \cdots$$

where

$$B_n = \{(b_1, b_2, \ldots, b_n) : b_i \in B \text{ and } b_1 \leq b_2 \leq \ldots \leq b_n\} .$$

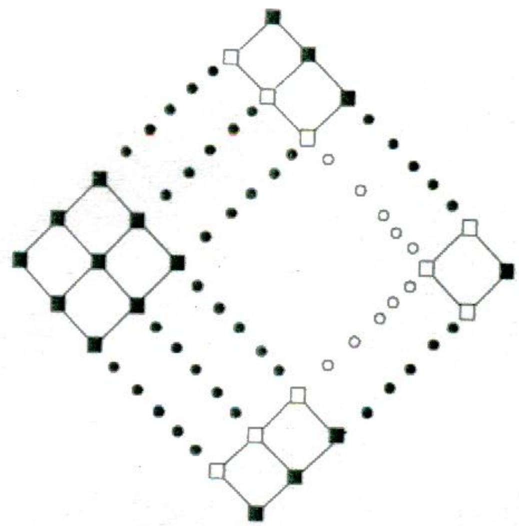
3. Let D be the lattice $D = \mathbb{N}^{\text{op}} \times \mathbf{2}$, where $\mathbb{N}$ is the set of natural numbers and $\mathbf{2} = \{0, 1\}$ and let B be the four element Boolean algebra $\{0, 1, b, b^*\}$. The lattice D has two non-principal filters, namely D and $\{(n, 1) : n \in \mathbb{N}\}$. Let $\varphi : B \to \mathcal{F}(D)^\partial$ be the map given by $\varphi(b) = \{(1, 0), (1, 1)\}$, $\varphi(b^*) = \{(n, 1) : n \in \mathbb{N}\}$, $\varphi(1) = \{(1, 1)\}$, and $\varphi(0) = D$. Note that $\varphi(b)$ is principal and $\varphi(b^*)$ is not. In general, if D does not have a 0, not both $\varphi(b)$ and $\varphi(b^*)$ can be principal. By Theorem 14, since B is finite, there are embeddings $S \subseteq \overline{S} \subseteq \overline{\overline{S}}$. These embeddings for $S = \mathcal{S}(B, D, \varphi)$ are depicted in the diagram below, with S black, $\overline{S}$ black and white, and $\overline{\overline{S}}$ black, white and gray.



4. In this example we have a pair of non-principal complementary filters in the image of φ. Let $X = \{1/i : i \in \mathbb{Z}\}$ and

$$\begin{aligned} x_i &= \{x \in X : -1/i \leq x\} \text{ for } i \in \mathbb{N}, \\ y_i &= \{x \in X : x \leq 1/i\} \text{ for } i \in \mathbb{N}. \end{aligned}$$

Then $x_i \cup y_j = X$ for all $i, j \in \mathbb{N}$. Let D be the lattice generated by $\{x_i, y_i : i \in \mathbb{N}\}$ in 2^X. Then $f = \{x_i : i \in \mathbb{N}\}$, $g = \{y_i : i \in \mathbb{N}\}$ are filters of D and $f \cap g = \{X\} = 0_{\mathcal{F}(D)}$, $f \vee g = D = 1_{\mathcal{F}(D)}$. Following is a diagram of $\overline{S}$ where $S = \mathcal{S}(B, D, \varphi)$, B is the four element Boolean algebra $\{0, b, b^*, 1\}$ and φ is given by $\varphi(b) = f$, $\varphi(b^*) = g$.



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