A Note on Negations and Nilpotent t-norms

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Abstract

In this paper, we explore general relationships among negations, convex Archimedean nilpotent t-norms, and automorphisms of the unit interval $\mathbb{I}$. *Each nilpotent t-norm has a (strong) negation naturally associated with it, namely,* $\eta_{\triangle}(x) = \bigvee \{y \in [0,1] : x \triangle y = 0\}$. *The same negation is determined by the formula* $\eta_{\triangle}(x) = f^{-1}(f(0)/f(x))$ *where* f *is a (multiplicative) generating function for the t-norm* $\triangle$. *A system* $(\mathbb{I}, \triangle, \triangledown, *)$ *is called de Morgan if* $x \triangledown y = (x^* \triangle y^*)^*$; *Stone if* $x \triangle y = 0$ *if and only if* $y \leq x^*$, *and* $x^* \triangledown x^{**} = 1$; *and Boolean if it is both de Morgan and Stone. A system is shown to be Boolean if and only if* $* = \eta_{\triangle}$ *and* $x \triangledown y = \eta_{\triangle}(\eta_{\triangle}(x) \triangle \eta_{\triangle}(y))$. *We also look at de Morgan, weak Boolean and Stone systems on the lattice* $\mathbb{I}^{[2]} = \{(x,y) \in \mathbb{I} \times \mathbb{I} : x \leq y\}$ *and compare properties of related systems on* $\mathbb{I}$ *and on* $\mathbb{I}^{[2]}$.

KEYWORDS: *negation, nilpotent t-norm, de Morgan system, Boolean system, Stone system*

1 Introduction

Logical connectives on fuzzy sets arise from those on the algebra of truth values, which is often taken to be either the unit interval or the lattice of subintervals of the unit interval. In earlier papers [1, 2] we considered de Morgan systems that occur in this setting, focussing primarily on de Morgan systems with strict t-norms. Here we turn our attention to negations and nilpotent t-norms, and consider Boolean, de Morgan, and Stone systems on the bounded lattice $\mathbb{I} = ([0,1], \leq, 0, 1)$ and on the bounded lattice $\mathbb{I}^{[2]} = \left([0,1]^{[2]}, \leq\ , 0, 1\right)$ of intervals, where $[0,1]^{[2]} = \{(x,y) \in [0,1] : x \leq y\}$ and $(a,b) \leq (c,d)$ if and only if $a \leq c$ and $b \leq d$.

This study was motivated by the fact that if f is a multiplicative generator of a convex, Archimedean nilpotent t-norm, then $f^{-1}(f(0)/f(x))$ is a negation. This natural connection between nilpotent t-norms and negations gives rise to special algebraic systems which we call Boolean systems, being reminiscent of Boolean algebras. Thus the theory of de Morgan systems with nilpotent t-norms is considerably different from that with strict t-norms, where there is no naturally associated negation. Additional interest is spurred by the fact that the negation associated with a nilpotent t-norm turns out to be the pseudocomplement with respect to that t-norm. The study of these phenomena and their generalizations to the interval-valued case seemed to be of some interest and to merit further study.

Definition 1 *A commutative, associative binary operation $\triangle$ on $\mathbb{I}$ is a* ***convex, Archimedean t-norm*** *if the following conditions hold: (1) $1 \triangle x = x$ for all $x \in [0,1]$; (2) The operation $\triangle$ is increasing in each variable, that is, if x, y, x_1, $y_1 \in [0,1]^{[2]}$ with $x \leq x_1$ and $y \leq y_1$, then $x \triangle y \leq x_1 \triangle y_1$; (3) The operation $\triangle$ is Archimedean, that is, $x \triangle x < x$ for all $x \in (0,1)$; (4) The operation $\triangle$ is convex, that is, if $x \triangle y \leq c \leq x_1 \triangle y_1$, there is an r between x and x_1 and an s between y and y_1 such that $c = r \triangle s$.*

A commutative, associative binary operation $\bigtriangledown$ on $\mathbb{I}$ is a ***convex, Archimedean t-conorm*** *if the following conditions hold: (1) $0 \bigtriangledown x = x$ for all $x \in [0,1]$; (2) The operation $\bigtriangledown$ is increasing in each variable; (3) The operation $\bigtriangledown$ is Archimedean. (4) The operation $\bigtriangledown$ is convex.*

Remark 2 *The condition of convexity for an operation $\mathbb{I}^2 \to \mathbb{I}$ is equivalent to continuity of that binary operation in the usual topology on the unit interval.*

The following definitions for t-norm and t-conorm were developed in [2].

Definition 3 *A commutative, associative binary operation $\triangle$ on $\mathbb{I}^{[2]}$ is a* ***convex, Archimedean t-norm*** *if the following conditions hold: (1) $(1,1) \triangle x = x$ for all $x \in [0,1]^{[2]}$; (2) $x \triangle (y \vee z) = (x \triangle y) \vee (x \triangle z)$ and $x \triangle (y \wedge z) = (x \triangle y) \wedge (x \triangle z)$ for all $x, y, z \in [0,1]^{[2]}$; (3) $C \triangle C \subseteq C$, where $C = \{(c,c) : c \in [0,1]\}$; and (4) $(0,1) \triangle (a,b) = (0,b)$ for all $(a,b) \in [0,1]^{[2]}$.*

A commutative, associative binary operation $\bigtriangledown$ on $\mathbb{I}^{[2]}$ is a ***convex, Archimedean t-conorm*** *if the following conditions hold: (1) $(0,0) \bigtriangledown x = x$ for all $x \in [0,1]^{[2]}$; (2) $x \bigtriangledown (y \wedge z) = (x \bigtriangledown y) \wedge (x \bigtriangledown z)$ and $x \bigtriangledown (y \vee z) = (x \bigtriangledown y) \vee (x \bigtriangledown z)$ for all $x, y, z \in [0,1]$; (3) $C \bigtriangledown C \subseteq C$, where $C = \{(c,c) : c \in [0,1]\}$; and (4) $(0,1) \bigtriangledown (a,b) = (a,1)$ for all $(a,b) \in [0,1]^{[2]}$.*

Definition 4 *For a lattice $\mathbb{L}$, an order-preserving [order-reversing] function $\mathbb{L} \to \mathbb{L}$ that is one-to-one and onto is an* ***automorphism*** *[respectively,* ***anti-automorphism****] of $\mathbb{L}$. An anti-automorphism η of $\mathbb{L}$ satisfying $\eta^2 = 1$ is called a* ***negation*** *(or strong negation), or an* ***involution****. We will denote the set of all automorphisms of $\mathbb{L}$ by $\mathrm{Aut}(\mathbb{L})$ and the set of all negations on $\mathbb{L}$ by $\mathrm{Neg}(\mathbb{L})$.*

In [2], we showed that every automorphism g of $\mathbb{I}^{[2]}$ is of the form $g(x,y) = (f(x), f(y))$ for an automorphism f of $\mathbb{I}$, and every anti-automorphism g of $\mathbb{I}^{[2]}$ is of the form $g(x,y) = (f(y), f(x))$ for an anti-automorphism f of $\mathbb{I}$. In particular, every negation on $\mathbb{I}^{[2]}$ is of the form $\eta(x,y) = (\beta(y), \beta(x))$ for β a negation on $\mathbb{I}$.

We denote the sets of all isomorphisms and anti-automorphisms of $\mathbb{I}$ and $\mathbb{I}^{[2]}$ by $\text{Map}(\mathbb{I})$ and $\text{Map}\left(\mathbb{I}^{[2]}\right)$, respectively. Both of these sets are groups under composition of maps and the relationship above preserves composition, i.e. $\text{Map}(\mathbb{I}) \approx \text{Map}\left(\mathbb{I}^{[2]}\right)$. Due to this isomorphism, many algebraic properties of the systems are similar. The logics associated with the two systems are intrinsically different, however [3, 6], and the de Morgan systems in these two settings differ algebraically.

If $\triangle$ is a t-norm and η a negation on $\mathbb{I}$, then it is well known that $\bigtriangledown$ defined by

$$x \bigtriangledown y = \eta(\eta(x) \triangle \eta(y))$$

is a t-conorm. It is easy to see that this holds for $\mathbb{I}^{[2]}$ as well.

Definition 5 *Let $\mathbb{L}$ be either $\mathbb{I}$ or $\mathbb{I}^{[2]}$. If a t-norm $\triangle$, a t-conorm $\bigtriangledown$, and a negation η satisfy the identity*

$$x \bigtriangledown y = \eta(\eta(x) \triangle \eta(y))$$

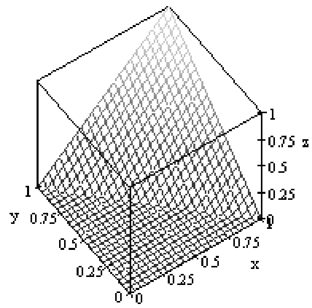
*for all $x, y \in \mathbb{L}$, then $(\mathbb{L}, \triangle, \bigtriangledown, \eta)$ is a **de Morgan system**, and the t-norm $\triangle$ and the t-conorm $\bigtriangledown$ are said to be **dual to one another** via the negation η.*

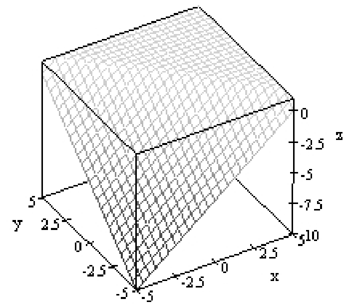
Definition 6 *A t-norm $\triangle$ is **nilpotent** if for each $x \in [0,1)$ there exists a positive integer n with $x \triangle x \triangle \cdots \triangle x = 0$ (n factors). A t-conorm is **nilpotent** if for each $x \in (0,1]$ there exists a positive integer n with $x \bigtriangledown x \bigtriangledown \cdots \bigtriangledown x = 1$ (n factors).*

Remark 7 *Throughout this paper, all t-norms and t-conorms are assumed to be convex, Archimedean, and (unless specifically mentioned otherwise) nilpotent. We will denote the set of all convex, Archimedean, nilpotent t-norms defined on $\mathbb{I}$ and $\mathbb{I}^{[2]}$ by* $\text{Nilp}(\mathbb{I})$ *and* $\text{Nilp}\left(\mathbb{I}^{[2]}\right)$.

The form of many of the results of this paper depends on our somewhat arbitrary choice of the Lukasiewicz t-norm, $x \blacktriangle y = (x + y - 1) \vee 0$, as a base point for the set of nilpotent t-norms and the corresponding negation α, defined by $\alpha(x) = 1 - x$, as a base point for the set of negations. Similar results hold for any compatible pair of choices of negation and nilpotent t-norm, although the notation is by far the simplest with the choice of α and $\blacktriangle$. These choices are related to setting the scaling for the lattice $\mathbb{I}$. The significance of these choices will be explored in a later paper on "averaging operators."

The following plots depict the nilpotent t-norm $x \blacktriangle y = (x + y - 1) \vee 0$ and the t-conorm $x \blacktriangledown y = (x + y) \wedge 1$ that is dual to $\blacktriangle$ via the negation α.





$x\blacktriangle y = (x+y-1) \vee 0$ $\qquad$ $x\blacktriangledown y = (x+y) \wedge 1$

2 Nilpotent t-norms and Negations on $\mathbb{I}$

In [1] we looked at maps

$$\begin{aligned} &\mathrm{Neg}\,(\mathbb{I}) \rightarrowtail \mathrm{Aut}\,(\mathbb{I}) : \eta \longmapsto \gamma_\eta : x \longmapsto \frac{\alpha\eta\,(x) + x}{2} \\ &\mathrm{Aut}\,(\mathbb{I}) \twoheadrightarrow \mathrm{Neg}\,(\mathbb{I}) : \gamma \longmapsto \alpha_\gamma = \gamma^{-1}\alpha\gamma \end{aligned} \tag{1}$$

and showed that the composition

$$\mathrm{Neg}\,(\mathbb{I}) \rightarrowtail \mathrm{Aut}\,(\mathbb{I}) \twoheadrightarrow \mathrm{Neg}\,(\mathbb{I}) : \eta \longmapsto \alpha_{\gamma_\eta}$$

is the identity map, inducing a one-to-one correspondence between $\mathrm{Neg}\,(\mathbb{I})$ and the set of right cosets of $\mathrm{Aut}\,(\mathbb{I})$ modulo the centralizer $Z\,(\alpha)$ of α. Note in particular that the map $\mathrm{Aut}\,(\mathbb{I}) \twoheadrightarrow \mathrm{Neg}\,(\mathbb{I})$ is onto. We also looked at the map from the set $\mathrm{Isom}\,(\mathbb{I}, \mathbb{I}_a)$ of bounded lattice isomorphisms to the set $\mathrm{Nilp}\,(\mathbb{I})$ of nilpotent t-norms

$$\mathrm{Isom}\,(\mathbb{I}, \mathbb{I}_a) \to \mathrm{Nilp}\,(\mathbb{I}) : f \longmapsto \triangle_f, \text{ where } x \triangle_f y = f^{-1}\left(f(x)f(y) \vee f(0)\right) \tag{2}$$

for $a \in (0,1)$, $\mathbb{I}_a = ([a,1], \vee, \wedge, a, 1)$. This is well known to be a one-to-one correspondence between the two sets. (See [1] for references and details.)

It is also generally known that isomorphisms $\mathbb{I} \cong \mathbb{I}_a$ lead to negations. In particular, given $f \in \mathrm{Isom}\,(\mathbb{I}, \mathbb{I}_a)$,

$$\eta(x) = f^{-1}\left(\frac{f(0)}{f(x)}\right)$$

is a negation. In fact, the converse holds as well. It is straightforward to check that $f\,(x) = e^{-\frac{\eta(x)+\alpha(x)}{2}}$ satisfies $f\eta\,(x) = f\,(0)\,/f\,(x)$. More generally, as a direct consequence of the fact that the map $\mathrm{Aut}\,(\mathbb{I}) \twoheadrightarrow \mathrm{Neg}\,(\mathbb{I})$ is onto, every negation η is of the form $\eta = \alpha_\gamma = \gamma^{-1}\alpha\gamma$ for some $\gamma \in \text{Aut}(\mathbb{I})$. Let $\varphi(x) = e^{x-1}$, and let $f = \varphi\gamma$. Then $\varphi^{-1}(x) = 1 + \ln x$ and

$$\begin{aligned} f^{-1}\left(\frac{f(0)}{f(x)}\right) &= \gamma^{-1}\left(\varphi^{-1}\left(\frac{\varphi(\gamma(0))}{\varphi(\gamma(x))}\right)\right) = \gamma^{-1}\left(1 + \ln\left(\frac{e^{-1}}{e^{\gamma(x)-1}}\right)\right) \\ &= \gamma^{-1}(1 - \gamma(x)) = \gamma^{-1}\alpha\gamma(x) = \alpha_\gamma(x) \end{aligned}$$

Note that each isomorphism $\varphi \in \text{Isom}(\mathbb{I}, \mathbb{I}_a)$ gives a one-to-one correspondence

$$\text{Aut}(\mathbb{I}) \to \text{Isom}(\mathbb{I}, \mathbb{I}_a) : \gamma \mapsto f = \varphi\gamma.$$

Each such φ gives a representation for a negation of the form

$$\alpha_{\varphi^{-1}f}(x) = f^{-1}\varphi\alpha\varphi^{-1}f(x)$$

that can be specialized not only to the above case but to others by making different choices for φ. This is illustrated in the following examples.

Example 8 *In the following, $0 \le x \le 1$, $0 < a < 1$, and f is an isomorphism $\mathbb{I} \approx \mathbb{I}_a$. For $\varphi(x) = a^{1-x}$,*

$$\alpha_{\varphi^{-1}f}(x) = f^{-1}\varphi\alpha\varphi^{-1}f(x) = f^{-1}\left(\frac{f(0)}{f(x)}\right)$$

For $\varphi(x) = (1-a)x + a$,

$$\alpha_{\varphi^{-1}f}(x) = f^{-1}\varphi\alpha\varphi^{-1}f(x) = f^{-1}(1 - f(x) + f(0))$$

For $\varphi(x) = (1-a)x^2 + a$

$$\begin{aligned} \alpha_{\varphi^{-1}f}(x) &= f^{-1}\varphi\alpha\varphi^{-1}f(x) \\ &= f^{-1}\left(1 + f(x) - f(0) + 2\sqrt{f(x) - f(0)f(x) - f(0) + f(0)^2}\right) \end{aligned}$$

A nilpotent (convex, Archimedean) t-norm is continuous and distributes over infinite meets and joins. We use these facts in the proof of the following theorem which reveals an important direct connection between nilpotent t-norms and negations.

Theorem 9 *A nilpotent t-norm $\triangle$ on $\mathbb{I}$ determines a negation $\eta_\triangle$ by the equation*

$$\eta_\triangle(x) = \bigvee \{y \in [0,1] : x \triangle y = 0\}$$

Proof. Since $0 \bigtriangleup y = 0$ for all y, $\eta_{\bigtriangleup}(0) = 1$; and since $1 \bigtriangleup y = y$ for all y, $\eta_{\bigtriangleup}(1) = 0$. If $x_1 \leq x_2$ then $x_1 \bigtriangleup y \leq x_2 \bigtriangleup y$, so that $\eta_{\bigtriangleup}(x_1) \geq \eta_{\bigtriangleup}(x_2)$. Now

$$\begin{aligned} x \bigtriangleup \eta_{\bigtriangleup}(x) &= x \bigtriangleup \left(\bigvee \{y \in [0,1] : x \bigtriangleup y = 0\} \right) \\ &= \left(\bigvee \{x \bigtriangleup y \in [0,1] : x \bigtriangleup y = 0\} \right) \\ &= 0 \end{aligned}$$

showing that

$$x \leq \eta_{\bigtriangleup}\left(\eta_{\bigtriangleup}(x)\right).$$

To see that $x = \eta_{\bigtriangleup}\left(\eta_{\bigtriangleup}(x)\right)$, we use the fact that $\bigtriangleup = \bigtriangleup_f$ for some isomorphism $f : [0,1] \to [f(0),1]$ with $0 < f(0) < 1$. Then

$$0 = x \bigtriangleup \eta_{\bigtriangleup}(x) = f^{-1}\left(f(x) f\left(\eta_{\bigtriangleup}(x)\right) \vee f(0)\right)$$

implies $f(x) f\left(\eta_{\bigtriangleup}(x)\right) \leq f(0)$. If $y > \eta_{\bigtriangleup}(x)$ then $x \bigtriangleup y > 0$ from which it follows that $f(x) f(y) > f(0)$. From the convexity of $\bigtriangleup$, we know that the function f is continuous, and thus that $f(x) f\left(\eta_{\bigtriangleup}(x)\right) = f(0)$. Now if $y > x$, $f(y) f\left(\eta_{\bigtriangleup}(x)\right) > f(x) f\left(\eta_{\bigtriangleup}(x)\right) = f(0)$. It follows that

$$y \bigtriangleup \eta_{\bigtriangleup}(x) = f^{-1}\left(f(y) f\left(\eta_{\bigtriangleup}(x)\right) \vee f(0)\right) = f^{-1}\left(f(y) f\left(\eta_{\bigtriangleup}(x)\right)\right) > 0$$

and thus that

$$x = \bigvee \left\{y \in [0,1] : y \bigtriangleup \eta_{\bigtriangleup}(x) = 0\right\} = \eta_{\bigtriangleup}\left(\eta_{\bigtriangleup}(x)\right).$$

■

Definition 10 *If $\bigtriangleup$ is a binary operation on a lattice $\mathbb{L}$ with 0, an element x^* in $\mathbb{L}$ is the $\bigtriangleup$-**pseudocomplement** of an element x if $x \bigtriangleup y = 0$ exactly when $y \leq x^*$.*

Theorem 9 says that for any nilpotent t-norm $\bigtriangleup$ on $\mathbb{I}$, the function $\eta_{\bigtriangleup}$ that gives the $\bigtriangleup$-pseudocomplement $\eta_{\bigtriangleup}(x)$ is a (strong) negation. It is easy to see that a t-norm $\bigtriangleup$ must be nilpotent in order for $\eta_{\bigtriangleup}$ to be a negation. As we shall soon see, every negation is the $\bigtriangleup$-pseudocomplement of some nilpotent t-norm $\bigtriangleup$. This will give two ways to represent negations—as $\alpha_{\gamma} = \gamma^{-1}\alpha\gamma$ for automorphisms γ of $\mathbb{I}$ (or more generally, as $\eta_{\gamma} = \gamma^{-1}\eta\gamma$ for fixed η), and as the $\bigtriangleup$-pseudocomplement $\eta_{\bigtriangleup}$ of a nilpotent t-norm $\bigtriangleup$. In order to show the connection between these two kinds of representations, we first look at a representation of nilpotent t-norms in terms of automorphisms of $\mathbb{I}$.

In [1] we showed that any two nilpotent t-norms $\circ$ and $\bigtriangleup$ determine isomorphic algebras $(\mathbb{I}, \circ)$ and $(\mathbb{I}, \bigtriangleup)$—that is, there is an automorphism γ of $\mathbb{I}$ such that $\gamma(x \circ y) = \gamma(x) \bigtriangleup \gamma(y)$ for all $x, y \in \mathbb{I}$. Moreover, given any convex, Archimedean, nilpotent t-norm $\bigtriangleup$, and any automorphism γ of $\mathbb{I}$, the binary operation $\bigtriangleup_{\gamma}$ defined by $x \bigtriangleup_{\gamma} y = \gamma^{-1}(\gamma(x) \bigtriangleup \gamma(y))$ is again a convex, Archimedean, nilpotent t-norm.

Example 11 *Take the t-norm $x\blacktriangle y=(x+y-1)\vee 0$. Then for any automorphism γ of $\mathbb{I}$, the binary operation*

$$x\blacktriangle_{\gamma}y=\gamma^{-1}\left((\gamma(x)+\gamma(y)-1)\vee 0\right)$$

is a convex, Archimedean, nilpotent t-norm. It follows from comments above that every convex, Archimedean, nilpotent t-norm is of this form for some automorphism γ of $\mathbb{I}$.

For any $0<a<1$, taking an arbitrary $\varphi, f\in \operatorname{Isom}(\mathbb{I},\mathbb{I}_a)$ and taking $x\blacktriangle y=(x+y-1)\vee 0$ gives a representation for a nilpotent t-norm of the form

$$x\blacktriangle_{\varphi^{-1}f}y=f^{-1}\varphi\left(\left(\varphi^{-1}f(x)+\varphi^{-1}f(y)-1\right)\vee\varphi^{-1}f\left(0\right)\right)$$

that can be specialized to familiar representations by making different choices for φ. Three such choices are illustrated in the following example.

Example 12 *In the following, $0<a<1$, $0\leq x\leq 1$, f is an isomorphism $\mathbb{I}\approx\mathbb{I}_a$, and $x\blacktriangle y=(x+y-1)\vee 0$. For $\varphi(x)=a^{1-x}$,*

$$x\blacktriangle_{\varphi^{-1}f}y=\ f^{-1}\left(f\left(x\right)f\left(y\right)\vee f\left(0\right)\right)$$

For $\varphi(x)=(1-a)\,x+a$,

$$x\blacktriangle_{\varphi^{-1}f}y=\ f^{-1}\left(\left(f\left(x\right)+f\left(y\right)-1\right)\vee f\left(0\right)\right)$$

For $\varphi(x)=(1-a)\,x^{2}+a$

$$\begin{aligned}x\blacktriangle_{\varphi^{-1}f}y=\ & f^{-1}((f\left(x\right)+f(y)+2(\sqrt{(f\left(x\right)-a)\left(1-a\right)}+\sqrt{(f(y)-a)\left(1-a\right)}\\ &-\sqrt{(f(x)-a)(f(y)-a)}\,)+1-2a)\vee f(0))\end{aligned}$$

We can now show the connection between the representations η_{γ} ($\gamma\in\operatorname{Aut}(\mathbb{I})$) and $\eta_{\triangle}$ ($\triangle\in\operatorname{Nilp}(\mathbb{I})$) for negations.

Theorem 13 *Choose any nilpotent t-norm $\boxtimes$ for a base point for $\operatorname{Nilp}(\mathbb{I})$ and let $\eta_{\boxtimes}$ be the $\boxtimes$-pseudocomplement. Then for each $\gamma\in\operatorname{Aut}(\mathbb{I})$*

$$\left(\eta_{\boxtimes}\right)_{\gamma}=\eta_{\boxtimes_{\gamma}}$$

Proof. Recall that $\left(\eta_{\boxtimes}\right)_{\gamma}=\gamma^{-1}\eta_{\boxtimes}\gamma$, $x\boxtimes_{\gamma}y=\gamma^{-1}\left(\gamma(x)\boxtimes\gamma(y)\right)$ and $\eta_{\boxtimes_{\gamma}}(x)=\bigvee\{y:x\boxtimes_{\gamma}y=0\}$. The equivalence of the following items is a direct consequence of the properties of γ as an isomorphism and the definitions of $\boxtimes_{\gamma}$ and $\eta_{\boxtimes}$.

$$\begin{aligned}x\boxtimes_{\gamma}y&=0\\ \gamma^{-1}\left(\left(\gamma(x)\boxtimes\gamma(y)\right)\right)&=0\\ \gamma(x)\boxtimes\gamma(y)&=0\\ \gamma(y)&\leq\eta_{\boxtimes}\left(\gamma(x)\right)\\ \gamma^{-1}\left(\gamma(y)\right)&\leq\gamma^{-1}\left(\eta_{\boxtimes}\left(\gamma(x)\right)\right)\\ y&\leq\gamma^{-1}\eta_{\boxtimes}\gamma(x)=\left(\eta_{\boxtimes}\right)_{\gamma}(x)\end{aligned}$$

It follows that

$$\left(\eta_{\boxtimes}\right)_{\gamma}(x)=\bigvee\left\{y: x \boxtimes_{\gamma} y=0\right\}=\eta_{\boxtimes_{\gamma}}(x)$$

Thus $\left(\eta_{\boxtimes}\right)_{\gamma}=\eta_{\boxtimes_{\gamma}}$. ■

This result is illustrated in the following examples.

Example 14 *For* $\varphi(x)=a^{x-1}$,

$$\begin{aligned}\left(\eta_{\blacktriangle}\right)_{\varphi^{-1} f}(x) &= f^{-1}\left(\frac{f(0)}{f(x)}\right) \\ &= \bigvee\left\{y \in[0,1]: f^{-1}(f(x) f(y) \vee f(0))=0\right\}=\eta_{\blacktriangle_{\varphi^{-1} f}}(x)\end{aligned}$$

and for $\varphi(x)=(1-a) x+a$,

$$\begin{aligned}\left(\eta_{\blacktriangle}\right)_{\varphi^{-1} f}(x) &= f^{-1}(1-f(x)+f(0)) \\ &= \bigvee\left\{y \in[0,1]: f^{-1}((f(x)+f(y)-1) \vee f(0))=0\right\}=\eta_{\blacktriangle_{\varphi^{-1} f}}(x)\end{aligned}$$

Representations of negations and nilpotent t-conorms are dual to those for t-norms.

Example 15 *In the following,* $0<a<1$, $0 \leq x \leq 1$, g *is an anti-isomorphism* $\mathbb{I} \approx \mathbb{I}_a$, *and* $x \blacktriangledown y=(x+y) \wedge 1$. *For* $\psi(x)=a^{-x}$,

$$\left(\eta_{\blacktriangledown}\right)_{\psi^{-1} g}(x)=g^{-1}\left(\frac{g(1)}{g(x)}\right)=\bigwedge\left\{y \in[0,1]: x \blacktriangledown_{\psi^{-1} g} y=1\right\}=\eta_{\blacktriangledown_{\psi^{-1} g}}(x)$$

and

$$x \blacktriangledown_{\psi^{-1} g} y=g^{-1}(g(x) g(y) \vee g(1))$$

For $\psi(x)=(a-1) x+1$,

$$\left(\eta_{\blacktriangledown}\right)_{\psi^{-1} g}(x)=g^{-1}(1-g(x)+a)=\bigwedge\left\{y \in[0,1]: x \blacktriangledown_{\psi^{-1} g} y=1\right\}=\eta_{\blacktriangledown_{\psi^{-1} g}}(x)$$

and

$$x \blacktriangledown_{\psi^{-1} g} y=g^{-1}((g(x)+g(y)-1) \vee g(1))$$

3 The Group of Nilpotent t-norms on $\mathbb{I}$

For each $\boxtimes \in \operatorname{Nilp}(\mathbb{I})$, the map $\operatorname{Aut}(\mathbb{I}) \longrightarrow \operatorname{Nilp}(\mathbb{I}): \gamma \mapsto \boxtimes_{\gamma}$ where $x \boxtimes_{\gamma} y=\gamma^{-1}(\gamma(x) \boxtimes \gamma(y))$ gives a one-to-one correspondence between the set of automorphisms of $\mathbb{I}$ and the set of nilpotent t-norms. If η is taken as base point for $\operatorname{Neg}(\mathbb{I})$, two different automorphisms σ and γ of $\mathbb{I}$ determine the same negation exactly when $\sigma \gamma^{-1}$ is in the centralizer $Z(\eta)$ of η, where $Z(\eta)=\{\sigma \in \operatorname{Aut}(\mathbb{I}): \sigma \eta=\eta \sigma\}$. The corresponding condition for nilpotent t-norms says that $\boxtimes_\sigma$ and $\boxtimes_\gamma$ determine the same negation exactly when $\sigma\gamma^{-1}$ is in the centralizer of $\eta_\boxtimes$.

This is summarized in the following commutative diagram

$$\begin{array}{ccccc} Z(\eta_\boxtimes) & \underset{\leftarrow}{\subseteq} & \mathrm{Aut}(\mathbb{I}) & \underset{\hookleftarrow}{\twoheadrightarrow} & \mathrm{Neg}(\mathbb{I}) \\ \approx\updownarrow & & \approx\updownarrow & & \| \\ \mathrm{Nilp}_\boxtimes(\mathbb{I}) & \underset{\leftarrow}{\subseteq} & \mathrm{Nilp}(\mathbb{I}) & \underset{\hookleftarrow}{\twoheadrightarrow} & \mathrm{Neg}(\mathbb{I}) \end{array} \tag{3}$$

where the vertical maps are one-to-one and onto, and

$$\mathrm{Nilp}_\boxtimes(\mathbb{I}) = \left\{ \triangle \in \mathrm{Nilp}(\mathbb{I}) : \eta_\triangle = \eta_\boxtimes \right\}.$$

These sequences "split"—that is, for the maps $\mathrm{Aut}(\mathbb{I}) \longrightarrow Z(\eta_\boxtimes) : \gamma \mapsto \frac{\eta_\boxtimes\gamma\eta_\boxtimes+\gamma}{2}$ and $\mathrm{Neg}(\mathbb{I}) \longrightarrow \mathrm{Aut}(\mathbb{I}) : \eta \mapsto \bar{\eta} : x \mapsto \frac{\eta_\boxtimes(\eta(x))+x}{2}$, the compositions $Z(\eta_\boxtimes) \subseteq \mathrm{Aut}(\mathbb{I}) \longrightarrow Z(\eta_\boxtimes)$ and $\mathrm{Neg}(\mathbb{I}) \longrightarrow \mathrm{Aut}(\mathbb{I}) \twoheadrightarrow \mathrm{Neg}(\mathbb{I})$ are the identity maps. (See [1] for details.) This does *not* split $\mathrm{Aut}(\mathbb{I})$ as a direct product of two subgroups, however, since $Z(\eta_\boxtimes)$ is not a normal subgroup for any $\boxtimes$.

Example 16 *The one-to-one correspondence*

$$\mathrm{Aut}(\mathbb{I}) \to \mathrm{Nilp}(\mathbb{I}) : \gamma \mapsto \blacktriangle_\gamma : x \blacktriangle_\gamma y = \gamma^{-1}\left((\gamma(x)+\gamma(y)-1) \vee 0\right)$$

endows the set of nilpotent t-norms with a group structure under the operation $\blacktriangle_\gamma \circ \blacktriangle_\delta = \blacktriangle_{\gamma\delta}$. *The subgroup corresponding to the group* $\mathbb{R}^+$ *of nonnegative reals is the one-parameter family of t-norms*

$$x \blacktriangle_r y = \sqrt[r]{(x^r + y^r - 1) \vee 0}$$

for $r > 0$.

4 Stone and Boolean Systems

The lattices $\mathbb{I}$ or $\mathbb{I}^{[2]}$ with the additional operations provided by a t-norm, t-conorm, and negation or other unary operation can satisfy properties reminiscent of axioms for de Morgan, Stone and Boolean algebras, and we name certain systems accordingly.

Definition 17 *Let* $\mathbb{L}$ *be either the lattice* $\mathbb{I}$ *or* $\mathbb{I}^{[2]}$ *with t-norm* $\triangle$, *t-conorm* $\bigtriangledown$, *and a decreasing unary operation* $*$. *We say that* $(\mathbb{L}, \triangle, \bigtriangledown, *)$ *is a* ***Stone system*** *if* $*$ *is a* $\triangle$*-pseudocomplement—that is,*

$$x \triangle y = 0 \text{ if and only if } y \leq x^*$$

and if $*$ *also satisfies the identity*

$$x^* \bigtriangledown x^{**} = 1$$

*for all elements x in the lattice. (In this case, $*$ is a $(\triangle, \bigtriangledown)$-**complement** on its image—that is, $x \triangle x^* = 0$ and $x \bigtriangledown x^* = 1$ for x in the image of $*$.) We say that $(\mathbb{L}, \triangle, \bigtriangledown, *)$ is a **weak Boolean** system if $*$ is a $\triangle$-pseudocomplement, and $(x \triangle y)^* = x^* \bigtriangledown y^*$ and $(x \bigtriangledown y)^* = x^* \triangle y^*$ for all $x, y \in \mathbb{L}$. We call $(\mathbb{L}, \triangle, \bigtriangledown, *)$ a **Boolean system** if it is both a Stone system and a de Morgan system.*

Remark 18 *The preceding definition applies to strict t-norms and t-conorms as well as to nilpotent ones, but the situation there is relatively trivial. If $\triangle$ is a strict t-norm, then the $\triangle$-pseudocomplement is given by $0^{*_\triangle} = 1$ and $x^{*_\triangle} = 0$ for $x \neq 0$, and for any t-conorm $\bigtriangledown$, $(\mathbb{L}, \triangle, \bigtriangledown, *_\triangle)$ is a Stone system. Note that the image of $*_\triangle$ is the two-element Boolean algebra, and on this image, $\triangle = \wedge$, $\bigtriangledown = \vee$ and $*_\triangle$ is the complement. Since this $\triangle$-pseudocomplement is not a negation, there are no Boolean systems with a strict t-norm.*

It is well known that a convex, Archimedean t-norm $\triangle$ on $\mathbb{I}$ is nilpotent exactly when there is a pair $x, y \in (0,1)$ with $x \triangle y = 0$, and a convex, Archimedean t-conorm $\bigtriangledown$ on $\mathbb{I}$ is nilpotent exactly when there is a pair $x, y \in (0,1)$ with $x \bigtriangledown y = 1$. This leads to the following necessary condition when $*$ is continuous.

Theorem 19 *If $(\mathbb{I}, \triangle, \bigtriangledown, *)$ is a Stone system in which $*$ is continuous, then both $\triangle$ and $\bigtriangledown$ are nilpotent.*

Proof. Let $a = \bigvee \{x : x^{**} = 0\}$ and $b = \bigwedge \{x : x^{**} = 1\}$ Then by continuity, $a^{**} = 0$ and $b^{**} = 1$. Also $0 \leq a < b \leq 1$. Let $a < x < b$. Then $x, x^*, x^{**} \in (0,1)$ so $x \triangle x^* = 0$ implies that $\triangle$ is nilpotent, and $x^* \bigtriangledown x^{**} = 1$ implies that $\bigtriangledown$ is nilpotent. ■

Theorem 9 established that for a nilpotent t-norm $\triangle$, the $\triangle$-pseudocomplement

$$\eta_\triangle(x) = \bigvee \{y \in [0,1] : x \triangle y = 0\}$$

is a negation. If η is any negation on $\mathbb{I}$ or $\mathbb{I}^{[2]}$ and $x \triangle \eta(x) = 0$, then the dual to $\triangle$ via the negation η satisfies

$$x \bigtriangledown \eta(x) = \eta(\eta(x) \triangle \eta(\eta(x))) = \eta(\eta(x) \triangle x) = \eta(0) = 1.$$

This yields the following theorems.

Theorem 20 *If $\triangle$ is a nilpotent t-norm, then $(\mathbb{I}, \triangle, \bigtriangledown, *)$ is a Boolean system if and only if $* = \eta_\triangle$ and $x \bigtriangledown y = \eta_\triangle(\eta_\triangle(x) \triangle \eta_\triangle(y))$.*

Theorem 21 *If $\triangle$ is a nilpotent t-norm, then $(\mathbb{I}, \triangle, \bigtriangledown, *)$ is a Stone system if and only if $* = \eta_\triangle$ and for all x,*

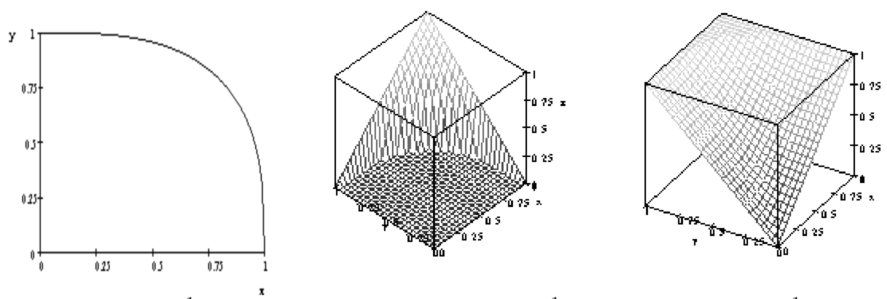
$$\eta_\triangle(x) \geq \eta_\bigtriangledown(x) = \bigwedge \{y \in [0,1] : x \bigtriangledown y = 1\}$$

If $(\mathbb{I}, \triangle, \bigtriangledown, *)$ is a Boolean system then equality holds (but this is not a sufficient condition since $\eta_\bigtriangledown$ does not determine $\bigtriangledown$).

In [1], we proved two basic facts about nilpotent convex Archimedean t-norms: any two are equivalent—that is, any two algebras $(\mathbb{I}, \triangle)$ are isomorphic, and each has a trivial automorphism group $\mathrm{Aut}\,(\mathbb{I}, \triangle)$. These two facts carry over immediately to Boolean systems, since the $\triangle$-pseudocomplement and the dual t-conorm are both naturally determined by $\triangle$.

Example 22 *The Boolean systems corresponding to the positive reals in the group* $\mathrm{Nilp}\,(\mathbb{I})$ *with base point* $\blacktriangle$ *are of the form* $\left(\mathbb{I}, \blacktriangle_r, \eta_{\blacktriangle_r}, \blacktriangledown_r\right)$ *with*

$$\begin{aligned} x \blacktriangle_r y &= ((x^r + y^r - 1) \vee 0)^{\frac{1}{r}} \\ \eta_{\blacktriangle_r} &= (1 - x^r)^{\frac{1}{r}} \\ x \blacktriangledown_r y &= (x^r + y^r)^{\frac{1}{r}} \wedge 1 \end{aligned}$$



$(1 - x^3)^{\frac{1}{3}}$ $\qquad$ $((x^3 + y^3 - 1) \vee 0)^{\frac{1}{3}}$ $\qquad$ $((x^3 + y^3))^{\frac{1}{3}} \wedge 1$

The one-parameter family $\blacktriangle_r$ of t-norms is well-known, and these t-norms are often paired with their duals relative to α. Members of the one-parameter family $\blacktriangledown_r$ of t-conorms are known as Yager t-conorms, the Yager t-norms being their duals relative to α.

Given a nilpotent t-norm there is exactly one Boolean system with this t-norm. Given a negation η, however, it follows from the commutative diagram (3) that the number of nilpotent t-norms $\triangle$ such that $\eta = \eta_{\triangle}$ is the same as the number of automorphisms in the centralizer of $\eta_{\triangle}$, which tells us how many different Boolean systems exist with a given η as the negation. Before making this count, we look at some specific examples of different Boolean systems having the same negation.

The functions of the form

$$\eta_{\lambda}(x) = \frac{1 - x}{1 + \lambda x}$$

for $\lambda > -1$ comprise a well-known family of negations, called *Sugeno negations*. In Section 2, we noted that $\eta(x) = f^{-1}(f(0)/f(x))$ for the exponential function $f(x) = \exp\left(-\frac{\eta(x)+\alpha(x)}{2}\right)$. In particular, the Sugeno negations are obtained using the functions $f_\lambda(x) = \exp\left(\frac{(-1+x)(2+\lambda x)}{2(1+\lambda x)}\right)$. The Sugeno negations are also generated by linear functions. For example, $\eta_\lambda(x) = \begin{cases} h_\lambda^{-1}\left(\frac{h_\lambda(0)}{h_\lambda(x)}\right) & \text{if } \lambda > 0 \\ h_\lambda^{-1}\left(\frac{h(1)}{h(x)}\right) & \text{if } -1 < \lambda < 0 \end{cases}$, where $h_\lambda(x) = \begin{cases} \dfrac{\lambda x + 1}{\lambda + 1} & \text{if } \lambda > 0 \\ \lambda x + 1 & \text{if } -1 < \lambda < 0 \end{cases}$. The two functions f_λ and h_λ generate *different* nilpotent t-norms, both of which induce the same Sugeno negation η_λ. To see this, we look at automorphisms γ_λ and σ_λ of $\mathbb{I}$ that generate the same t-norms and negations as f_λ and h_λ. Taking $\varphi(x) = (f_\lambda(0))^{1-x}$ and $\varphi(x) = (h_\lambda(0))^{1-x}$ in these two cases, gives the automorphisms

$$\gamma_\lambda(x) = \frac{(2+\lambda+\lambda x)\,x}{2(1+\lambda x)} \text{ for } \lambda > -1 \text{ and } \sigma_\lambda(x) = \begin{cases} \frac{\ln(1+\lambda x)}{\ln(1+\lambda)} & \text{if } \quad 0 < \lambda \\ 1 + \frac{\ln(1+\lambda x)}{\ln(1+\lambda)} & \text{if } \ -1 < \lambda < 0 \end{cases}$$

and the corresponding t-norms

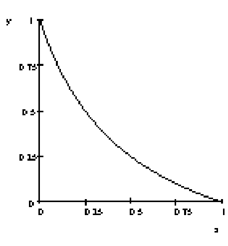
$$x \blacktriangle_{\sigma_\lambda} y = \frac{\lambda xy + x + y - 1}{1+\lambda} \vee 0 \quad \text{and} \quad x \blacktriangle_{\gamma_\lambda} y = \frac{p_\lambda(x,y) + \sqrt{q_\lambda(x,y)}}{2\lambda(1+\lambda x)(1+\lambda y)} \vee 0$$

where

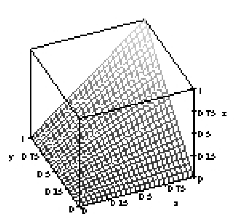
$$p_\lambda(x,y) = -2\lambda^2 y - 3\lambda - \lambda^3 xy - 2\lambda^2 x + 2\lambda^2 xy + \lambda^3 x^2 y + \lambda^2 x^2 + \lambda^2 y^2 + \lambda^3 xy^2 - 2$$

$$\begin{aligned} q_\lambda(x,y) &= 4 + 4(2x+2y+1)\lambda + \left(8x^2 + 24xy + 8y^2 - 4x - 4y + 9\right)\lambda^2 \\ &+ 2\left(2x^3 + 14x^2 y + 14xy^2 + 2y^3 - 5x^2 - 12xy - 5y^2 + 6x + 6y\right)\lambda^3 \\ &+ \left(\begin{array}{c} x^4 + 12x^3 y + 30x^2 y^2 + 12xy^3 + y^4 - 4x^3 - 22x^2 y \\ -22xy^2 - 4y^3 + 4x^2 + 14xy + 4y^2 \end{array}\right)\lambda^4 \\ &+ 2xy(x+y)\left(x^2 + 4xy + y^2 - 3x - 3y + 2\right)\lambda^5 + x^2 y^2 (x+y-1)^2 \lambda^6 \end{aligned}$$

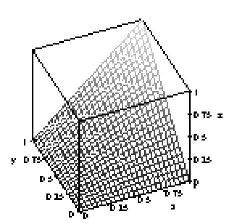
As evidence that these two t-norms are not the same, we give a plot of their difference for $\lambda = 2$.



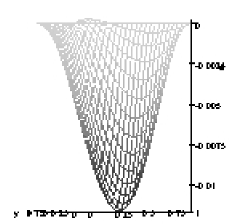
$\eta_2(x) = \dfrac{1-x}{1+2x}$



$x \blacktriangle_{\gamma_2} y$



$x \blacktriangle_{\sigma_2} y$



$x \blacktriangle_{\gamma_2} y - x \blacktriangle_{\sigma_2} y$

This gives two different Boolean systems $\left(\mathbb{I}, \blacktriangle_{\gamma_\lambda}, \eta_\lambda, \blacktriangledown_{\gamma_\lambda}\right)$ and $(\mathbb{I}, \blacktriangle_{\sigma_\lambda}, \eta_\lambda, \blacktriangledown_{\sigma_\lambda})$ for each λ. The Sugeno negation $\eta_2(x)$, which the two t-norms have in common as $\bigvee \{y : x \bigtriangleup y = 0\}$, can be observed in the two t-norm plots above as the boundary of the zero set.

To establish the size of the centralizer of a negation, we consider the following (where we assume α as the base point for the set of negations). Two different automorphisms σ and γ of $\mathbb{I}$ determine the same negation $\beta^{-1}\alpha\beta = \gamma^{-1}\alpha\gamma$ exactly when $\sigma\gamma^{-1}$ is in the centralizer of α. The corresponding condition for isomorphisms $f = \varphi\gamma, h = \varphi\sigma : \mathbb{I} \to \mathbb{I}_a$ is that $fg^{-1} \in Z(\varphi\alpha\varphi^{-1})$, where $Z(\varphi\alpha\varphi^{-1})$ denotes the centralizer of $\varphi\alpha\varphi^{-1}$ in $\operatorname{Aut}(\mathbb{I}_a)$. Pick $a \in (0,1)$, and let $\varphi(x) = a^{1-x}$. Then $\varphi\alpha\varphi^{-1}(x) = \frac{a}{x}$—that is, $\varphi\alpha\varphi^{-1} = a\iota^{-1}$. It is easy to check that the maps in the centralizer of $a\iota^{-1}$ are exactly the set

$$Z\left(a\iota^{-1}\right) = \left\{ s \in \operatorname{Aut}(\mathbb{I}_a) : s\left(\frac{a}{x}\right) = \frac{a}{s(x)} \text{ for all } x \in [a,1] \right\}.$$

This subgroup of $\operatorname{Aut}(\mathbb{I}_a)$ is isomorphic to the group $\operatorname{Aut}([a, \sqrt{a}])$ under the restriction map. To see this, observe that if $s\left(\frac{a}{x}\right) s(x) = a$ then $s\left(\frac{a}{\sqrt{a}}\right) s(\sqrt{a}) = (s(\sqrt{a}))^2 = a$, so that $s(\sqrt{a}) = \sqrt{a}$. This gives a map $\rho : Z(a\iota^{-1}) \to \operatorname{Aut}([a, \sqrt{a}])$ with $\rho(s)$ the automorphism obtained from s by restricting the domain of s to $[a, \sqrt{a}]$. The map ρ is clearly a monomorphism since each s is completely determined by its behavior on $[a, \sqrt{a}]$. To see that ρ is onto, let $k : [a, \sqrt{a}] \to [a, \sqrt{a}]$ be any order isomorphism. Define $s_k : [a,1] \to [a,1]$ by

$$s_k(x) = \begin{cases} k(x) & if \quad a \le x \le \sqrt{a} \\ \frac{a}{k\left(\frac{a}{x}\right)} & if \quad \sqrt{a} \le x \le 1 \end{cases}$$

It is easy to check that $s_k \in Z(a\iota^{-1})$ and $\rho(s_k) = k$, so the map ρ is onto and hence is an isomorphism. Finally, observe that $Z(\alpha) \to Z(\varphi\alpha\varphi^{-1}) : \beta \mapsto \varphi^{-1}\beta\varphi$ is one-to-one and onto. We conclude that $Z(\alpha)$ has the power of the continuum and, in particular, for each negation η there are the power of the continuum different t-norms $\bigtriangleup$ having η as $\bigtriangleup$-pseudocomplement.

Corollary 23 *For each negation η there are uncountably many Boolean systems $(\mathbb{I}, \bigtriangleup, \eta, \bigtriangledown)$.*

We now turn to systems on $\mathbb{I}^{[2]}$. In [2] we showed that convex, Archimedean t-norms $\bigtriangleup$ and t-conorms $\bigtriangledown$ on $\mathbb{I}^{[2]}$ are of the form

$$(a,b) \bigtriangleup (c,d) = (a \circ c, b \circ d) \;\text{ and }\; (a,b) \bigtriangledown (c,d) = (a \diamond c, b \diamond d)$$

where $\circ$ is a t-norm on $\mathbb{I}$ and $\diamond$ is a t-conorm on $\mathbb{I}$.

Lemma 24 *Let $\triangle$ be a nilpotent t-norm on $\mathbb{I}^{[2]}$ and let $\circ$ be the t-norm on $\mathbb{I}$ such that $(a,b) \triangle (c,d) = (a \circ c, b \circ d)$. Then the $\triangle$-pseudocomplement of (a,b) in $\mathbb{I}^{[2]}$ is*

$$(a,b)^{*\triangle} = (\eta_\circ(b), \eta_\circ(b))$$

where $\eta_\circ(b)$ is the $\circ$-pseudocomplement of b in $\mathbb{I}$.

Theorem 25 *A nilpotent system $(\mathbb{I}^{[2]}, \triangle, \bigtriangledown, *)$ is a Stone system if and only if $* = *_\triangle$ and the $\bigtriangledown$-pseudocomplement defined by $x^{*\bigtriangledown} = \bigwedge \left\{ y \in \mathbb{I}^{[2]} : x \bigtriangledown y = 1 \right\}$ satisfies*

$$(b,b)^{*\triangle} \geq (b,b)^{*\bigtriangledown}$$

for all $(b,b) \in \mathbb{I}^{[2]}$.

Proof. Suppose $(\mathbb{I}^{[2]}, \triangle, \bigtriangledown, *)$ is a Stone system, with $(a,b)\triangle(c,d) = (a \circ c, b \circ d)$. Now $(a,b)\triangle(c,d) = (0,0)$ if and only if $(c,d) \leq (a,b)^*$. But $(a \circ c, b \circ d) = (0,0)$ if and only if $c \leq \eta_\circ(a)$ and $d \leq \eta_\circ(b)$. Since $c \leq d$ and $\eta_\circ(b) \leq \eta_\circ(a)$, this is equivalent to $(c,d) \leq (\eta_\circ(b), \eta_\circ(b))$. It follows that $(a,b)^* = (\eta_\circ(b), \eta_\circ(b)) = (a,b)^{*\triangle}$. So $* = *_\triangle$.

Also

$$(1,1) = ((b,b)^{*\triangle})^{*\triangle} \bigtriangledown (b,b)^{*\triangle} = (b,b) \bigtriangledown (b,b)^{*\triangle}$$

Now $(b,b) \bigtriangledown (c,d) = (1,1)$ if and only if $(c,d) \geq (b,b)^{*\bigtriangledown}$, so $(b,b)^{*\triangle} \geq (b,b)^{*\bigtriangledown}$.

For the converse, it is easy to see that $*_\triangle$ is the pseudocomplement. Also note that $(a,b)^{*\triangle} = (b,b)^{*\triangle}$ for all $a \leq b$. Assume that $(b,b)^{*\triangle} \geq (b,b)^{*\bigtriangledown}$ for all $b \in [0,1]$. Then

$$((a,b)^{*\triangle})^{*\triangle} \bigtriangledown (a,b)^{*\triangle} = (b,b) \bigtriangledown (b,b)^{*\triangle} \geq (b,b) \bigtriangledown (b,b)^{*\bigtriangledown} = (1,1).$$

This completes the proof. ∎

Note that $*_\triangle$ is not a negation, but is the unique $\triangle$-pseudocomplement for $\mathbb{I}^{[2]}$, so in particular, a Stone system on $\mathbb{I}^{[2]}$ is never Boolean.

Theorem 26 *Suppose $\left(\mathbb{I}^{[2]}, \triangle, \bigtriangledown, *\right)$ is a Stone system, and let $(\mathbb{I}, \circ, \diamond, \eta_\circ)$ be the system on $\mathbb{I}$ satisfying*

$$\begin{aligned} (a,b) \triangle (c,d) &= (a \circ c, b \circ d) \\ (a,b) \bigtriangledown (c,d) &= (a \diamond c, b \diamond d) \\ (a,b)^* &= (\eta_\circ(b), \eta_\circ(b)) \end{aligned}$$

Then $(\mathbb{I}, \circ, \diamond, \eta_\circ)$ is a Stone system. Moreover, $\left(\mathbb{I}^{[2]}, \triangle, \bigtriangledown, *\right)$ is a weak Boolean system if and only if the corresponding system $(\mathbb{I}, \circ, \diamond, \eta_\circ)$ is a Boolean system.

Proof. By Theorem 25, $\eta_\circ$ is the $\circ$-pseudocomplement for the nilpotent t-norm $\circ$ on $\mathbb{I}$. Also

$$\begin{aligned} (a,b)^{*\triangle} &= (\eta_\circ(b), \eta_\circ(b)) \\ &\geq (a,b)^{*\bigtriangledown} = (\eta_\diamond(b), \eta_\diamond(b)) \end{aligned}$$

so $\eta_\circ(b) \geq \eta_\diamond(b)$ for all $b \in I$, so $(\mathbb{I}, \circ, \diamond, \eta_\circ)$ is a Stone system. Now for $x, y \in \mathbb{I}^{[2]}$

$$(x \bigtriangleup y)^* = ((a,b) \bigtriangleup (c,d))^* = (a \circ c, b \circ d)^* = (\eta_\circ(b \circ d), \eta_\circ(b \circ d))$$

and

$$x^* \bigtriangledown y^* = (a,b)^* \bigtriangledown (c,d)^* = (\eta_\circ(b), \eta_\circ(b)) \bigtriangledown (\eta_\circ(d), \eta_\circ(d)) = (\eta_\circ(b) \diamond \eta_\circ(d), \eta_\circ(b) \diamond \eta_\circ(d))$$

If $(\mathbb{I}, \circ, \diamond, \eta_\circ)$ is a Boolean system, then $\eta_\circ(b) \diamond \eta_\circ(d) = \eta_\circ(b \circ d)$ and $\eta_\circ(b) \circ \eta_\circ(d) = \eta_\circ(b \diamond d)$, so

$$\begin{aligned} x^* \bigtriangledown y^* &= (\eta_\circ(b) \diamond \eta_\circ(d), \eta_\circ(b) \diamond \eta_\circ(d)) = (\eta_\circ(b \circ d), \eta_\circ(b \circ d)) = (x \bigtriangleup y)^* \\ x^* \bigtriangleup y^* &= (\eta_\circ(b) \circ \eta_\circ(d), \eta_\circ(b) \circ \eta_\circ(d)) = (\eta_\circ(b \diamond d), \eta_\circ(b \diamond d)) = (x \bigtriangledown y)^* \end{aligned}$$

On the other hand, if $\left(\mathbb{I}^{[2]}, \bigtriangleup, \bigtriangledown, *\right)$ is a weak Boolean system, then

$$(\eta_\circ(b) \diamond \eta_\circ(d), \eta_\circ(b) \diamond \eta_\circ(d)) = x^* \bigtriangledown y^* = (x \bigtriangleup y)^* = (\eta_\circ(b \circ d), \eta_\circ(b \circ d))$$

Since $\eta_\circ$ is a negation, this implies the system $(\mathbb{I}, \circ, \diamond, \eta_\circ)$ is a de Morgan system, and hence a Boolean system. ■

Note that each nilpotent t-norm on $\mathbb{I}^{[2]}$ determines a unique weak Boolean system on $\mathbb{I}^{[2]}$. Note also that a Stone system on $\mathbb{I}^{[2]}$ satisfying either of the two conditions $x^* \bigtriangledown y^* = (x \bigtriangleup y)^*$ or $x^* \bigtriangleup y^* = (x \bigtriangledown y)^*$ satisfies both and thus is a weak Boolean system.

Let $\left(\mathbb{I}^{[2]}, \bigtriangleup, \bigtriangledown, *\right)$ be any Stone system on $\mathbb{I}^{[2]}$. The analogy with Stone algebras is apparent in the following observations that are reminiscent of the triple construction (see [4]). The image of $*$ is the sublattice $\{(c,c) : c \in [0,1]\}$ which is isomorphic to $\mathbb{I}$ and is a Boolean system under the induced operations. The kernel of $*$ is the sublattice $\{(a,1) : a \in [0,1]\}$. Every element of $\mathbb{I}^{[2]}$ is of the form $(a,b) = (c,1) \bigtriangleup (b,b)$ since $0 \circ b \leq a \leq 1 \circ b$ implies $a = c \circ b$ for some c.

Summary: Every convex, Archimedean nilpotent t-norm $\bigtriangleup$ is isomorphic to the Lukasiewicz t-norm $\blacktriangle$, that is, for some automorphism γ of the unit interval $\mathbb{I} = ([0,1], \leq)$, and all $x, y \in [0,1]$,

$$\gamma(x \bigtriangleup y) = \gamma(x) \blacktriangle \gamma(y) = (\gamma(x) + \gamma(y) - 1) \vee 0$$

This isomorphism is unique, thus endowing the set of all nilpotent t-norms with a group structure.

Each nilpotent t-norm $\bigtriangleup$ on $\mathbb{I}$ determines a negation $\eta_\bigtriangleup$ by the condition $\eta_\bigtriangleup(x)$ is maximal such that $x \bigtriangleup \eta_\bigtriangleup(x) = 0$. This negation is also determined by γ, namely, $\eta_\bigtriangleup = \gamma^{-1}\alpha\gamma$ where $\alpha(x) = 1 - x$, as well as by any multiplicative generator f of the t-norm as $f^{-1}(f(0)/f(x))$. For any given negation η, there are uncountably many nilpotent t-norms $\bigtriangleup$ such that $\eta = \eta_\bigtriangleup$.

The lattices $\mathbb{I}$ or $\mathbb{I}^{[2]}$ with the additional operations provided by a t-norm, t-conorm, and negation or other unary operation can satisfy properties reminiscent of axioms for de Morgan, Stone and Boolean algebras, and we named certain systems accordingly. We identified the Stone and Boolean systems $(\mathbb{I}, \triangle, \triangledown, *)$ over $\mathbb{I}$—the Boolean systems turn out to be those isomorphic to the de Morgan system $(\mathbb{I}, \blacktriangle, \alpha)$, known to generate the variety of MV-algebras. We identified the Stone systems $\left(\mathbb{I}^{[2]}, \triangle, \triangledown, *\right)$ and showed there are no Boolean systems on $\mathbb{I}^{[2]}$. Moreover, $\left(\mathbb{I}^{[2]}, \triangle, \triangledown, *\right)$ is a weak Boolean system if and only if it induces a Boolean system on $\mathbb{I}$.

The Boolean systems on $\mathbb{I}$ and Stone systems on $\mathbb{I}^{[2]}$ have special features as a result of the interplay between the t-norm and the negation. In the literature, de Morgan systems commonly have nilpotent t-norms paired with t-conorms that are dual via $1 - x$, even when $1 - x$ is not the negation naturally associated with the t-norm. We suggest that, since the Boolean and Stone systems are natural in theory, they may also be useful in applications.

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