GROUPS, T-NORMS, AND FAMILIES OF DE MORGAN SYSTEMS

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1 Introduction

Continuous Archimedean t-norms are generated by automorphisms of the unit interval with its usual order structure. Certain subgroups of the group A of automorphisms of the unit interval—the group operation being composition of functions—play an important role in the theory, for example the multiplicative group of positive real numbers, which is embedded in A by $r(x) = x^r$. Several standard families of t-norms are in natural one-to-one correspondence with subgroups of A. We examine this phenomenon, and various other group theoretic aspects of t-norm theory.

Operations on fuzzy sets are defined in terms of their membership functions, and as such, are basically operations on the unit interval. The standard operations suggested by Zadeh in 1965 [15] included minimum and maximum for "and" and "or", and the function $1 - x$ for "complement." He also suggested the algebraic product xy as an alternative for "and." These are not the only ways to extend classical set theory consistently, and many people have suggested alternatives to these fuzzy set theoretic operations. A general definition for t-norms (and), t-conorms (or), and negations (not) has evolved, and several families of t-norms are now regularly referred to: Hamacher, Dombi, Yager, Schweizer, and Frank, to name a few. Such families are typically one-parameter families that are in one-to-one correspondence with some subset of the real numbers, such as the positive ones. The Hamacher family [8]

$$\left\{ \frac{xy}{a + (1 - a)(x + y - xy)} : a > 0 \right\}$$

is one example. These and other continuous Archimedean t-norms can be obtained from automorphisms of the lattice $([0, 1], \leq)$. For the Hamacher family of t-norms, this family of automorphisms is

$$\left\{ \frac{x}{x + a(1 - x)} : a > 0 \right\}$$

That is, the automorphism defined by $f_a(x) = x/(x + a(1-x))$ generates the corresponding Hamacher t-norm in the sense that

$$\frac{xy}{a + (1-a)(x + y - xy)} = f_a^{-1}(f_a(x) f_a(y))$$

This family $\{f_a : a > 0\}$ of automorphisms is closed under composition and forms a subgroup of the group A of all automorphisms of the ordered unit interval. The principal fact that we want to point out is that several of the well known families of t-norms come from just a few simply expressed subgroups of A, or cosets or conjugates of such subgroups. Having generators expressed as compositions of simply expressed automorphisms makes some computations easier, and suggests new families. We provide a number of examples, and consider nilpotent as well as strict t-norms.

2 Definitions and notation

Triangular norms (t-norms) are central items of study in fuzzy set theory. A **t-norm** is a binary operation $\triangle$ on $\mathbb{I} = ([0,1], \leq)$, the unit interval with its usual order relation, such that, for all x, y, $z \in [0,1]$,

1. $1 \triangle x = x$
2. $x \triangle y = y \triangle x$
3. $(x \triangle y) \triangle z = x \triangle (y \triangle z)$
4. The operation $\triangle$ is increasing in each variable—that is, $x \leq x_1$ and $y \leq y_1$ imply that $x \triangle y \leq x_1 \triangle y_1$.

A dual notion to t-norm is that of **t-conorm**, a binary operation $\bigtriangledown$ on $\mathbb{I}$ such that, for all x, y, $z \in [0,1]$,

1. $0 \bigtriangledown x = x$
2. $x \bigtriangledown y = y \bigtriangledown x$
3. $(x \bigtriangledown y) \bigtriangledown z = x \bigtriangledown (y \bigtriangledown z)$
4. The operation $\bigtriangledown$ is increasing in each variable.

A t-norm is **convex** if whenever $x \triangle y \leq c \leq x_1 \triangle y_1$, then there is an r between x and x_1 and an s between y and y_1 such that $c = r \triangle s$. This condition is equivalent to continuity in each variable. We write $a^{[n]}$ for $a \triangle a \triangle \cdots \triangle a$, the t-norm of a with itself n times. A t-norm $\triangle$ is **Archimedean** if for each $a, b \in (0,1)$, there is a positive integer n such that $a^{[n]} < b$.

A t-norm $\triangle$ is **nilpotent** if it is convex, Archimedean, and for $a \neq 1$, $a^{[n]} = 0$ for some positive integer n, the n depending on a. Those convex Archimedean t-norms that are not nilpotent are called **strict**. Multiplication xy is the generic example of a strict t-norm, and $(x + y - 1) \vee 0$ the generic example of a nilpotent one.

An **automorphism** of $\mathbb{I}$ is a one-to-one mapping of $[0, 1]$ onto $[0, 1]$ such that $f(a) \leq f(b)$ if and only if $a \leq b$. An **anti-automorphism** of $\mathbb{I}$ is a one-to-one mapping g of $[0, 1]$ onto $[0, 1]$ such that $g(a) \geq g(b)$ if and only if $a \leq b$. The set M of all automorphisms and anti-automorphisms of $\mathbb{I}$ is a group under composition of maps, and the set A of all automorphisms is a subgroup of index 2 in M. A copy of the multiplicative group of positive real numbers is contained in A by associating a positive real number r with the mapping $[0, 1] \to [0, 1] : x \to x^r$. This association is a homomorphism mapping the group of positive real numbers isomorphically onto a subgroup of A. This subgroup will be denoted by $\mathbb{R}^+$. If f is any automorphism of $\mathbb{I}$, rf will denote the composition of functions—that is, $rf(x) = (f(x))^r$. Anti-automorphisms in M of order two are called **negations**, or **involutions**. They will play an important role in what we do.

The connection between strict t-norms and the group A is this [9]:

Theorem 1 An Archimedean t-norm $\triangle$ is strict if and only if there is an automorphism $f \in A$ such that $f(x \triangle y) = f(x) f(y)$. Another automorphism $g \in A$ satisfies this condition if and only if $f = rg$ for some $r \in R^+$ (where r denotes the function defined by $r(x) = x^r$).

Such an automorphism f is called a (multiplicative) **generator** of the t-norm $\triangle$ since $x \triangle y = f^{-1}(f(x)f(y))$, that is, the strict t-norm $\triangle$ is determined by f and ordinary multiplication.

The connection between nilpotent t-norms and A is the following (see [6, 9]).

Theorem 2 A t-norm $\triangle$ is nilpotent if and only if there is an automorphism $f \in A$ such that $f(x \triangle y) = (f(x) + f(y) - 1) \vee 0$. The automorphism f is unique.

The automorphism f is called an **L-generator** of the nilpotent t-norm $\triangle$ since $x \triangle y = f^{-1}((f(x) + f(y) - 1) \vee 0)$, that is, the nilpotent t-norm $\triangle$ is determined by f and the Łukasiewicz t-norm $(x + y - 1) \vee 0$ [7].

In subsequent sections we describe several connections between families of t-norms and groups of generating functions. These connections are generally obscured by the common use of "additive generators" for triangular norms, as the collection of additive generators does not form a group in a natural way. Also, multiplicative and L-generators are isomorphisms between the algebras $(\mathbb{I}, \circ)$ and $(\mathbb{I}, \triangle)$, where $\circ$ represents either multiplication or the Łukasiewicz t-norm, and $\triangle$ a strict or nilpotent t-norm, respectively. Additive generators do not play such a natural algebraic role.

We refer to [2, 3] for additional details.

3 Subgroups of the automorphism group

Theorem 2 puts the set of nilpotent t-norms in one-to-one correspondence with the automorphism group A. This places a group structure on the set of all nilpotent t-norms for which the Łukasiewicz t-norm acts as the identity element.

For strict t-norms, the situation is a bit different. Strict t-norms are in one-to-one correspondence with right cosets of $\mathbb{R}^+$ in A, where the multiplicative group $\mathbb{R}^+$ of positive real numbers is identified with a subgroup of A by $r(x) = x^r$ for each positive r. Multiplication of elements of $\mathbb{R}^+$ corresponds to composition of functions in A. Were $\mathbb{R}^+$ a normal subgroup in A, this would put a natural group structure on the set of all strict t-norms. This is not the case, however. That is, the normalizer $\{f \in A : f^{-1}\mathbb{R}^+ f = \mathbb{R}^+\}$ of $\mathbb{R}^+$ in the group A is not all of A. But the set of t-norms with generators in the normalizer of $\mathbb{R}^+$ in the group A does carry a group structure. This group has been identified and its structure determined [11]. The corresponding set of t-norms is the one-parameter family of Aczél-Alsina t-norms [1]. Details also appear in [10], Section 5.8.1. We describe those results briefly.

3.1 The normalizer of the subgroup of positive reals

Let N be the **normalizer** of $\mathbb{R}^+$ in the group M; that is,

$$N = \left\{ f \in M : f^{-1}\mathbb{R}^+ f = \mathbb{R}^+ \right\}.$$

The following hold for the group N.

1. $N = \left\{ f \in M : f(x) = e^{-c(-\ln x)^r},\ c > 0,\ r \neq 0 \right\}$

2. N is isomorphic to the group of pairs of real numbers

$$\{(c, r) : c > 0, r \neq 0\}$$

   with multiplication given by

$$(c, r)(d, s) = (cd^r, rs).$$

   The subgroup $\mathbb{R}^+ = \{e^{-c(-\ln x)} : c > 0\}$ corresponds to the group $\{(c, 1) : c \in \mathbb{R}^+\}$, and the subgroup $S = \{e^{-(-\ln x)^r} : r \neq 0\}$ to the group $\{(1, r) : r \neq 0\}$. Thus N **splits** over $\mathbb{R}^+$—that is, $\mathbb{R}^+$ is normal in N and any element of N is uniquely a product rf, $r \in \mathbb{R}^+$, $f \in S$. The group S is called the **Richman group**. Notice that elements of the Richman group S fix $\frac{1}{e}$.

3. The t-norms with generators in N are given by

$$x \circ_r y = e^{-((-\ln x)^r + (-\ln y)^r)^{\frac{1}{r}}}$$

with r positive. These are **Aczél-Alsina** t-norms [1]. The t-conorms with generators in N are given by

$$x \diamond_r y = e^{-((-\ln x)^r + (-\ln y)^r)^{\frac{1}{r}}}$$

with r negative. The group S gives the same family of t-norms and t-conorms since an element of N is of the form rf with $r \in \mathbb{R}^+$ and $f \in S$.

4. The negations in N are precisely the elements $\eta_c = e^{-c(-\ln x)^{-1}} = e^{\frac{c}{\ln x}}$—that is, the elements in N with parameter $r = -1$.

5. The t-norm $x \circ_r y = e^{-((-\ln x)^r + (-\ln y)^r)^{\frac{1}{r}}}$ is dual to the t-conorm $x \diamond_s y = e^{-\left((-\ln x)^{-s} + (-\ln y)^s\right)^{\frac{1}{-s}}}$ $(r > 0, s < 0)$ if and only if $r = -s$, in which case they are dual with respect to precisely the negations $e^{\frac{c}{\ln x}}$ in N. (A t-norm and negation, together with the dual t-conorm, is called a **De Morgan system** or **triple system**. The normalizer of $\mathbb{R}^+$ gives a two-parameter family of De Morgan systems for $s > 0$, $c > 0$.)

Fixing the parameter c at $c = 1$ gives a one-parameter family of isomorphic De Morgan systems.

Theorem 3 The De Morgan system $(\mathbb{I}, \circ_r, \eta_1)$ is isomorphic to the De Morgan system $(\mathbb{I}, \cdot, \eta_1)$ for all $r > 0, r \neq 0$. In particular, the De Morgan systems $(\mathbb{I}, \circ_r, \eta_1)$ are isomorphic to one another for all $r > 0$.

Proof. Let $h(x) = e^{-(-\ln x)^r}$, $\eta_1(x) = e^{\frac{1}{\ln x}}$ and $x \circ_r y = e^{-((-\ln x)^r + (-\ln y)^r)^{\frac{1}{r}}}$ Then

$$\begin{aligned} h(x \circ_r y) &= \exp\left(-\left[\left[(-\ln x)^r + (-\ln y)^r\right]^{\frac{1}{r}}\right]^r\right) \\ &= \exp\left(-\left((-\ln x)^r + (-\ln y)^r\right)\right) = h(x)h(y). \end{aligned}$$

Also,

$$\begin{aligned} h(\eta_1(x)) = h\left(e^{\frac{1}{\ln x}}\right) &= \exp\left(-\left(-\ln\left(e^{\frac{1}{\ln x}}\right)\right)^r\right) \\ &= \exp\left(-(-\ln x)^{r-1}\right) = \exp\left(\left(\ln\left(e^{-(-\ln x)^r}\right)\right)^{-1}\right) \\ &= \exp\left((-\ln h(x))^{-1}\right) = \eta_1(h(x)). \end{aligned}$$

Thus h is an isomorphism between the two systems. ■

3.2 Some special generator functions

Many of the well known families of t-norms (along with some other interesting, but not so well known, families) are very closely connected with the group N and its subgroups $\mathbb{R}^+$ and S. We will express the sets of generators of these families of t-norms as simple combinations of these three subgroups of M, two special functions in M, and one special subset of M. These are the following:

- the subgroup $\mathbb{R}^+$ of A;
- the subgroup $N = \{e^{-c(-\ln x)^r} : c > 0,\ r \neq 0\}$ of M;
- the subgroup $S = \{e^{-(-\ln x)^r} : r \neq 0\}$ of N;
- the function $\alpha(x) = 1 - x$ in M;
- the function $\varepsilon(x) = e^{-\frac{1-x}{x}}$ in A;
- The set $F = \{\varphi_a(x) = \frac{a^x - 1}{a - 1} : a > 0,\ a \neq 1\} \subset A$.

These groups and functions are closely connected. It was mentioned earlier that $N = \mathbb{R}^+ \times S$. An easy computation shows that $\varepsilon\alpha\varepsilon^{-1}(x) = e^{\frac{1}{\ln x}}$ so that $\varepsilon\alpha\varepsilon^{-1} \in S$, and in particular, ε and $\varepsilon\alpha$ determine the same right cosets of S and of N. That is, $S\varepsilon = S\varepsilon\alpha$ and $N\varepsilon = N\varepsilon\alpha$. This comes into play in the discussion below of the Dombi t-norms. Also, in some cases, α gives the duality between pairs of t-norms and t-conorms in a family even though α is not a member of the generating group or coset.

Another family of subgroups that come into play are the centralizers of negations, in particular, the centralizer of the negation α:

$$Z(\alpha) = \{z \in A : z\alpha = \alpha z\}$$

You may recognize F as the set of generators of the Frank family of t-norms. The subset F is not a subgroup of A under composition. It is, however, closely related to the subgroups $\varepsilon^{-1}\mathbb{R}^+\varepsilon$ and $Z(\alpha)$ as follows. For each $r \in \mathbb{R}^+$, $\varepsilon^{-1}r^{-1}\varepsilon\varphi_{r^2} \in Z(\alpha)$. This implies that $\varepsilon^{-1}r\varepsilon$ and φ_{r^2} represent the same left coset of $Z(\alpha)$ as φ_{r^2}—that is, $\varepsilon^{-1}r\varepsilon Z(\alpha) = \varphi_{r^2} Z(\alpha)$. Moreover, $\varepsilon^{-1}\mathbb{R}^+\varepsilon \cap Z(\alpha) = \{1\}$. This follows easily from the fact that $z\left(\frac{1}{2}\right) = \frac{1}{2}$ for all $z \in Z(a)$. The set F is otherwise an anomaly in this setting. In particular, it is not closed under composition.

Now we list some families of t-norms (and t-conorms in some cases), and as promised, express their sets of generators in terms of the subgroups, subsets, and functions mentioned above.

4 Families of strict t-norms

1. The Hamacher family (t-norms and generators) [8]:

   $$\left\{\frac{xy}{x+y-xy+a(1-x-y+xy)} : a > 0\right\}$$

   $$\left\{\frac{x}{x+a(1-x)} : a > 0\right\} = \varepsilon^{-1}\mathbb{R}^{+}\varepsilon$$

   Comments: The group $\mathbb{R}^{+}$ gives only one t-norm, namely multiplication. However, the group $\varepsilon^{-1}\mathbb{R}^{+}\varepsilon$, which is a conjugate of $\mathbb{R}^{+}$, gives a one-parameter family of t-norms. Since the group $\varepsilon^{-1}\mathbb{R}^{+}\varepsilon$ contains only automorphisms, it does not generate any t-conorms.

2. The Jane Doe #2 family (t-norms and generators):

   $$\left\{xye^{-a\ln x\ln y} : a > 0\right\}$$

   $$\left\{\frac{1}{1-\ln x^{a}} : a > 0\right\} = \varepsilon^{-1}\mathbb{R}^{+}$$

   Comments: The coset $\varepsilon^{-1}\mathbb{R}^{+}$ gives a one-parameter family of t-norms. Since the coset contains only automorphisms, it does not generate any t-conorms.

3. The Jane Doe #3 family (t-norms and generators):

   $$\left\{1-(1-(1-(1-x)^{a})(1-(1-y)^{a}))^{\frac{1}{a}} : a > 0\right\}$$

   $$\{1-(1-x)^{a} : a > 0\} = \alpha\mathbb{R}^{+}\alpha$$

   Comments: The group $\mathbb{R}^{+}$ gives only one t-norm, namely multiplication. However, the group $\alpha\mathbb{R}^{+}\alpha$, which is a conjugate of $\mathbb{R}^{+}$, gives a one-parameter family of t-norms. Since the group $\alpha\mathbb{R}^{+}\alpha$ contains only automorphisms, it does not generate any t-conorms.

4. The Schweizer family (t-norms and generators) [12]:

   $$\left\{(x^{-a}+y^{-a}-1)^{-\frac{1}{a}} : a > 0\right\}$$

   $$\left\{e^{-\left(\frac{1-x^{a}}{x^{a}}\right)} : a > 0\right\} = \varepsilon\mathbb{R}^{+}$$

Comments: The coset $\varepsilon\mathbb{R}^+$ gives a one-parameter family of t-norms. Since the coset contains only automorphisms, it does not generate any t-conorms.

5. The Aczél-Alsina family (t-norms, t-conorms and generators) [1]:

$$\left\{e^{-((-\ln x)^r+(-\ln y)^r)^{\frac{1}{r}}} : r > 0\right\}$$

$$\left\{e^{-((-\ln x)^r+(-\ln y)^r)^{\frac{1}{r}}} : r < 0\right\}$$

$$\left\{e^{-(-\ln x)^r} : r \neq 0\right\} = S$$

Comments: The only negation in this group of generators is the generator with $r = -1$, which gives the negation $e^{1/\ln x}$. This group gives a family of De Morgan systems, namely the t-norm with parameter r, $r > 0$, the negation $e^{1/\ln x}$, and the t-conorm with parameter $-r$.

6. The Jane Doe #1 family (t-norms, t-conorms and generators):

$$\left\{\left(1+\left[(\tfrac{1-x}{x})^r+(\tfrac{1-y}{y})^r+(\tfrac{1-x}{x})^r(\tfrac{1-y}{y})^r\right]^{\frac{1}{r}}\right)^{-1} : r > 0\right\}$$

$$\left\{\left(1+\left[(\tfrac{1-x}{x})^r+(\tfrac{1-y}{y})^r+(\tfrac{1-x}{x})^r(\tfrac{1-y}{y})^r\right]^{\frac{1}{r}}\right)^{-1} : r < 0\right\}$$

$$\left\{\frac{x^r}{x^r+(1-x)^r} : r \neq 0\right\} = \varepsilon^{-1}S\varepsilon$$

Comments on this family: The only negation in this group of generators is the generator with $r = -1$, which gives the negation α. This group gives a family of De Morgan systems, namely the t-norm with parameter r, $r > 0$, the negation $\alpha(x) = \varepsilon^{-1}\left(e^{\frac{1}{\ln \varepsilon(x)}}\right) = 1-x$, and the t-conorm with parameter $-r$. The family of generators in this example is a conjugate of the group S of generators of the Aczél-Alsina family. Any conjugate of S will give a family of De Morgan systems having the same algebraic properties as those of the Aczél-Alsina family.

7. The Jane Doe #1-Hamacher family (t-norms, t-conorms and generators):

$$\left\{\left(1+[(\tfrac{1-x}{x})^r+(\tfrac{1-y}{y})^r+a((\tfrac{1-x}{x})^r(\tfrac{1-y}{y})^r]^{\frac{1}{r}}\right)^{-1} : a>0, r>0\right\}$$

$$\left\{\left(1+[(\tfrac{1-x}{x})^r+(\tfrac{1-y}{y})^r+a((\tfrac{1-x}{x})^r(\tfrac{1-y}{y})^r]^{\frac{1}{r}}\right)^{-1} : a>0, r<0\right\}$$

$$\left\{\frac{x^r}{x^r+a(1-x)^r} : a>0,\ r\neq 0\right\} = \varepsilon^{-1}N\varepsilon$$

Comments: The negations in this group of generators are those with parameters $a>0$, $r=-1$, which are the negations $(1-x)/(1+(a-1)x)$. This group gives a family of De Morgan systems, namely the t-norm with parameters $a>0$, $r>0$, the negation $(1-x)/(1+(a-1)x)$, and the t-conorm with parameters a, $-r$.

8. The Dombi family (t-norms, t-conorms and generators) [4]:

$$\left\{\left(1+\left(\tfrac{1-x}{x}\right)^r+\left(\tfrac{1-y}{y}\right)^r\right)^{-\frac{1}{r}} : r>0\right\}$$

$$\left\{\left(1+\left(\tfrac{1-x}{x}\right)^r+\left(\tfrac{1-y}{y}\right)^r\right)^{-\frac{1}{r}} : r<0\right\}$$

$$\left\{e^{-\left(\frac{1-x}{x}\right)^r} : r\neq 0\right\} = S\varepsilon$$

Comments: The coset $S\varepsilon$ has no negation in it. The t-norm with parameter r ($r>0$), is dual to the t-conorm with parameter $-r$, with respect to the negation α. Denote the element $e^{-\left(\frac{1-x}{x}\right)^r}$ of $S\varepsilon$ by $t_r\varepsilon$. Then the element $\varepsilon\alpha\varepsilon^{-1} = e^{1/\ln x} = t_{-1}\in S$. For a generator $t_r\varepsilon$ of a t-norm in the Dombi family, $t_r\varepsilon\alpha = t_r\varepsilon\alpha\varepsilon^{-1}\varepsilon = t_r t_{-1}\varepsilon \in S\varepsilon$.

9. The Frank family (t-norms and generators) [5]:

$$\left\{\log_a\left(1+\tfrac{(a^x-1)(a^y-1)}{a-1}\right) : a>0,\ a\neq 1\right\}$$

$$\left\{\tfrac{a^x-1}{a-1} : a>0,\ a\neq 1\right\} = F$$

Comments: The set of Frank t-norms is the set of t-norms $\triangle$ satisfying the equation $x\triangle y + x\triangledown y = x+y$, where $\triangledown$ is the t-conorm dual to $\triangle$ with respect to the negation α. This set of t-norms does not fit the pattern of the other well-known families of strict t-norms, as it does not come from a group or a coset of a group of generators. Nevertheless, the negation α plays a critical role in its defining equation. The dual t-conorms for the Frank family relative to α and their generators are

$$\left\{1-\log_a\left(1+\frac{\left(a^{1-x}-1\right)\left(a^{1-y}-1\right)}{a-1}\right):a>0,a\neq 1\right\}$$

$$\left\{\frac{a^{1-x}-1}{a-1}:a>0,\ a\neq 1\right\}=F\alpha$$

This family can be reparametrized as follows: For each $a>1$, $a\neq 1$, write $a=e^{-b}$. Then for $b>0$, $e^{-b}>0$ and $e^{-b}\neq 1$. So replacing a by e^{-b} reparametrizes this family with the parameters the positive reals. This makes the parameters into a group under multiplication. However, this operation is not related to the group operation of composition of the generators.

There does not appear to be a direct group theoretic representation of this family, but as pointed out in Section 3.2, it has close ties to the subgroup $Z(\alpha)$, the centralizer of α.

5 Families of nilpotent t-norms

1. The Schweizer-Sklar family (t-norms and L-generators) [12]:

$$\left\{((x^a+y^a-1)\vee 0)^{\frac{1}{a}}:a>0\right\}$$

$$\{x^a:a>0\}=\mathbb{R}^+$$

Comments on this family: The set of generators of this family forms a group, inducing a group structure on the set of Schweizer-Sklar t-norms. The identity of this group is the Łukasiewicz t-norm $(x+y-1)\vee 0$, corresponding to the parameter $a=1$.

2. The Yager family (t-norms and L-generators) [14]:

$$\left\{(1-((1-x)^a+(1-y)^a)^{\frac{1}{a}})\vee 0:a>0\right\}$$

$$\{1-(1-x)^a:a>0\}=\alpha^{-1}\mathbb{R}^+\alpha$$

Comments on this family: The set $\alpha^{-1}\mathbb{R}^+\alpha$ of generators forms a group, being a conjugate of $\mathbb{R}^+$. The Schweizer-Sklar and Yager families have only the Łukasiewicz t-norm in common, corresponding to the fact $\mathbb{R}^+\cap\alpha^{-1}\mathbb{R}^+\alpha=\{1\}$.

3. The Weber family (t-norms and L-generators) [13]:

$$\{(a(x+y-1)-(a-1)xy)\vee 0 : a>0\}$$

$$\{1-\log_a(1-(1-a)(1-x)) : a>0, a\neq 1\} = \alpha^{-1}F^{-1}\alpha$$

Comments on this family: This set of generators does not form a group and is not a coset of a subgroup of M. The generators of this family, and the next two, are related to the generators of the Frank family of strict t-norms or t-conorms.

4. Jane Doe #4 (t-norms and L-generators):

$$\left\{\left(\tfrac{1}{a}\left(x+y-1+(a-1)xy\right)\right)\vee 0 : a>0,\ a\neq 1\right\}$$

$$\{\log_a(1+(a-1)x) : a>0,\ a\neq 1\} = F^{-1}$$

5. Jane Doe #6 (t-norms and L-generators):

$$\left\{\log_a\left(1+\tfrac{|a-1|}{a-1}\left(\left(\tfrac{|a-1|}{a-1}\left(a^x+a^y-a-1\right)\right)\vee 0\right)\right) : a>0, a\neq 1\right\}$$

$$\left\{\frac{a^x-1}{a-1} : a>0, a\neq 1\right\} = F$$

6 Summary Comments

Just why the generators of so many of the well known families of strict and nilpotent t-norms are simply group theoretic combinations—for example, cosets or conjugates of the groups $\mathbb{R}^+$, S, and N—is not too clear. It is of interest to note that the groups S and N come from $\mathbb{R}^+$ rather directly, N being the normalizer of $\mathbb{R}^+$ and S being a complementary factor to $\mathbb{R}^+$ in N, as indicated in Section 3. As mentioned in Section 3.2, the Frank family (and consequently the Weber, Jane Doe #4, and Jane Doe #6 nilpotent families with related generating sets) does not come from a subgroup, but the set of Frank family generators is closely related to the subgroups $\varepsilon^{-1}\mathbb{R}^+\varepsilon$ and $Z(\alpha)$.

References

[1] J. Aczél and C. Alsina, *Characterizations of some classes of quasilinear functions with applications to triangular norms and to synthesizing judgements*, Aequationes Mathematicae **25** (1982), 313-315.

[2] Mai Gehrke, Carol Walker, and Elbert Walker, *De Morgan Systems on the Unit Interval*, International Journal of Intelligent Systems **11** (1996), 733-750.

[3] Mai Gehrke, Carol Walker, and Elbert Walker, *A note on negations and nilpotent t-norms*, International Journal of Approximate Reasoning **21** (1999), 137-155.

[4] J. Dombi, *A general class of fuzzy operators, the De Morgan class of fuzzy operators and fuzziness measures induced by fuzzy operators*, Fuzzy Sets and Systems **8** (1982), 149-163.

[5] M. J. Frank, *On the simultaneous associativity of* $F(x, y)$ *and* $x + y - F(x, y)$, Aequationes Math. **19** (1979), 194-226 .

[6] Mai Gehrke, Carol Walker, and Elbert Walker, *A Note on Negations and Nilpotent t-norms*, International Journal of Approximate Reasoning **21** (1999), 137-155.

[7] R. Giles, *Łukasiewicz logic and fuzzy theory*, Inter. J. Man-Mach. Stud. **8** (1976), 313-327.

[8] H. Hamacher, *Über logische Aggregationen nicht-binär expliziter Entscheidungskriterien*, Frankfurt/Main (1978).

[9] C. H. Ling, *Representation of Associative Functions*, Publ. Math. Debrecen **12** (1963), 182-212.

[10] Hung Nguyen and Elbert Walker, *A First Course in Fuzzy Logic*, Second Edition, Chapman & Hall/CRC Press, Boca Raton (2000).

[11] Fred Richman and Elbert Walker, *Some Group Theoretic Aspects of t-norms*, International Journal of Uncertainty, Fuzziness, and Knowledge-based Systems **8** (2000), 1-6.

[12] B. Schweizer and A. Sklar, *Associative functions and statistical triangle inequalities*, Publicationes Mathematicae Debrecen **8** (1961), 169-186.

[13] S. Weber, *A general concept of fuzzy connectives, negations, and implications based on t-norms and t-conorms*, Fuzzy Sets Syst. **11** (1983), 115-134.

[14] R. R. Yager, *On a general class of fuzzy connectives*, Fuzzy Sets and Systems, **4** (1980) 235-242.

[15] Zadeh, L. A., Fuzzy Sets, *Information and Control*, **8** (1965) 338-353.

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