Some Group Theoretic Aspects of t-norms

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Abstract

Continuous archimedean t-norms are generated by automorphisms f of $([0,1],\leq)$, the unit interval with its usual order structure. Such strict t-norms $\triangle$ are given by $x\triangle y=f^{-1}(f(x)f(y))$ and nilpotent ones by $x\triangle y=f^{-1}((f(x)+f(y)-1)\vee 0)$. Let A be the group of automorphisms of $([0,1],\leq)$, the group operation being composition of functions. Certain subgroups of A play an important role in the theory, for example the multiplicative group $\mathbb{R}^+$ of positive real numbers, which is embedded in A by $r(x)=x^r$. Some standard families of t-norms are in natural one-to-one correspondence with subgroups of A. We examine this phenomenon, and various other group theoretic aspects of t-norm theory.

1 Introduction

There are several families of t-norms regularly referred to: Hamacher, Dombi, Yager, Schweizer, and Frank, to name a few. Now t-norms come from automorphisms of the lattice $([0,1],\leq)$. Such families are typically one-parameter families, that is, are in one-to-one correspondence with some subset of the real numbers, for example with the positive ones. The Hamacher family $\{\frac{xy}{a+(1-a)(x+y-xy)} : a>0\}$ is a good example, and the corresponding family of automorphisms is $\{\frac{x}{x+a(1-x)} : a>0\}$. That is, $f(x)=\frac{x}{x+a(1-x)}$ generates $x\triangle y=\frac{xy}{a+(1-a)(x+y-xy)}$ in the sense that $f^{-1}(f(x)f(y))=x\triangle y$. This family of automorphisms is a subgroup of the group A of all automorphisms of $([0,1],\leq)$. The principal fact that we want to point out is that several of these well known families come from just a few simply expressed subgroups of A, or cosets or conjugates of such groups. Having generators expressed as compositions of simply expressed automorphisms makes some computations easy, and suggests new families. We provide a number of examples, and consider nilpotent as well as strict t-norms.

2 Definitions and Notation

Triangular norms (t-norms) are binary operations on the unit interval $[0,1]$ and are central items of study in fuzzy set theory. Let $\mathbb{I}=([0,1],\leq)$, the unit interval with its usual order relation. An **automorphism** of $\mathbb{I}$ is a one-to-one mapping of $\mathbb{I}$ onto $\mathbb{I}$ such that $f(a)\leq f(b)$ if and only if $a\leq b$. An **anti-automorphism** of $\mathbb{I}$ is a one-to-one mapping g of $\mathbb{I}$ onto $\mathbb{I}$ such that $g(a)\geq g(b)$ if and only if $a\leq b$. The set M of all automorphisms and anti-automorphisms of $\mathbb{I}$ is a group under composition of maps, and the set A of all automorphisms is a subgroup of index 2 in M. A copy of the multiplicative group of positive real numbers is contained in A by associating a positive real number r with the mapping $[0,1]\to[0,1]:x\to x^r$. This subgroup will be denoted by $\mathbb{R}^+$. Anti-automorphisms in M of order two are called **negations**, or **involutions**. They will play an important role in what we do.

A **t-norm** is a binary operation $\triangle$ on $[0,1]$ such that, for all x, y, $z\in[0,1]$,

1. $1\triangle x=x$
2. $x\triangle y=y\triangle x$
3. $(x\triangle y)\triangle z=x\triangle(y\triangle z)$
4. The operation $\triangle$ is increasing in each variable–that is, $x\leq x_1$ and $y\leq y_1$ imply that $x\triangle y\leq x_1\triangle y_1$.

A t-norm is **convex** if whenever $x\triangle y\leq c\leq x_1\triangle y_1$, then there is an r between x and x_1 and an s between y and y_1 such that $c=r\triangle s$. This condition is equivalent to continuity in each variable. We write $a^{[n]}$ for $a\triangle a\triangle\cdots\triangle a$, the t-norm of a with itself n times. A t-norm $\triangle$ is **Archimedean** if for each $a,b\in(0,1)$, there is a positive integer n such that $a^{[n]}<b$.

A t-norm $\triangle$ is **nilpotent** if it is convex, Archimedean, and for $a\neq 1$, $a^{[n]}=0$ for some positive integer n, the n depending on a. Those convex

Archimedean t-norms that are not nilpotent are called **strict**. Multiplication xy is the generic example of a strict t-norm, and $(x+y-1)\vee 0$ the generic example of a nilpotent one.

A dual notion is that of **t-conorm.** We refer to [1] for details. The connection between strict t-norms and the group A is this [3]:

Theorem 1 *An Archimedean t-norm $\triangle$ is strict if and only if there is an element $f \in A$ such that $f(x \triangle y) = f(x) f(y)$. Another element $g \in A$ satisfies this condition if and only if $f = rg$ for some $r \in \mathbb{R}^+$ (where r denotes the function defined by $r(x) = x^r$).*

Such an element f is called a **generator** of the t-norm $\triangle$ since $x \triangle y = f^{-1}(f(x)f(y))$, that is, $\triangle$ is determined by f and ordinary multiplication.

The connection between nilpotent t-norms and A is this [2].

Theorem 2 *A t-norm $\triangle$ is nilpotent if and only if there is an element $f \in A$ such that $f(x \triangle y) = (f(x) + f(y) - 1) \vee 0$. The element f is unique.*

The element f is called an **L-generator** of the nilpotent t-norm $\triangle$.

3 Some Preliminary Results

The previous theorem says that nilpotent t-norms are in one-to-one correspondence with the group A. For strict t-norms, the situation is a bit different. Strict t-norms are in one-to-one correspondence with right cosets of $\mathbb{R}^+$ in A, where the multiplicative group $\mathbb{R}^+$ of positive real numbers is identified with a subgroup of A by $r(x) = x^r$ for each positive r. Multiplication of elements of $\mathbb{R}^+$ corresponds to composition of functions in A. Were $\mathbb{R}^+$ a normal subgroup in A, this would put a natural group structure on the set of all strict t-norms. This is not the case, however. That is, the normalizer $\left\{f \in A : f^{-1}\mathbb{R}^+ f = \mathbb{R}^+\right\}$ of $\mathbb{R}^+$ in the group A is not all of A. But the set of t-norms with generators in the normalizer of $\mathbb{R}^+$ in the group A does carry a group structure. This group has been identified, its structure determined, and explicit formulas given for the corresponding set of t-norms [4]. We describe those results briefly.

Let N be the **normalizer** of $\mathbb{R}^+$ in the group M; that is,

$$N = \left\{f \in M : f^{-1}\mathbb{R}^+ f = \mathbb{R}^+\right\}.$$

The following hold for the group N.

1. $N = \left\{f \in M : f(x) = e^{-c(-\ln x)^r},\ c > 0,\ r \neq 0\right\}$
2. N is isomorphic to the group of pairs of real numbers
$$\{(c, r) : c > 0, r \neq 0\}$$
with multiplication given by
$$(c, r)(d, s) = (cd^r, rs).$$
The subgroup $\mathbb{R}^+ = \{e^{-c(-\ln x)} : c > 0\}$ corresponds to the group $\{(c, 1) : c \in \mathbb{R}^+\}$, and the subgroup $R = \{e^{-(-\ln x)^r} : r \neq 0\}$ to the group $\{(1, r) : r \neq 0\}$. Thus N **splits** over $\mathbb{R}^+$, that is, $\mathbb{R}^+$ is normal in N and any element of N is uniquely a product xy, $x \in \mathbb{R}^+$, $y \in R$. The group R is called the **Richman group**. Notice that the Richman group R fixes $\frac{1}{e}$.
3. The t-norms with generators in N are given by
$$x \triangle y = e^{-((-\ln x)^r + (-\ln y)^r)^{\frac{1}{r}}}$$
with r positive. The t-conorms with generators in N are given by
$$x \triangledown y = e^{-((-\ln x)^r + (-\ln y)^r)^{\frac{1}{r}}}$$
with r negative. These are **Aczél-Alsina** t-norms and t-conorms. The group R would give the same family of t-norms and t-conorms since an element of N is of the form rf with $r \in \mathbb{R}^+$ and $f \in R$.
4. The negations in N are precisely the elements $e^{-c(-\ln x)^{-1}} = e^{\frac{c}{\ln x}}$, that is, the elements in N with parameter $r = -1$.
5. The t-norm $e^{-((-\ln x)^s + (-\ln y)^s)^{\frac{1}{s}}}$ $(s, t > 0)$ is dual to the t-conorm $e^{-\left((-\ln x)^{-t} + (-\ln y)^{-t}\right)^{\frac{1}{-t}}}$ if and only if $s = t$, in which case they are dual with respect to precisely the negations $e^{\frac{c}{\ln x}}$ in N. (A t-norm and negation, together with the dual t-conorm, is called a De Morgan system or triple system. The normalizer of $\mathbb{R}^+$ gives a two-parameter family of De Morgan systems for $s > 0$, $c > 0$.)

It turns out that many of the well known families of t-norms (along with some other interesting, but not so well known, families) are very closely connected with the group N and its subgroups $\mathbb{R}^+$ and R. We are going to express the set of generators of these families of t-norms as simple combinations of these three subgroups of M, two special elements of M, and one special subset of M. These are the following:

- the subgroup $\mathbb{R}^+$ of A;
- the subgroup $N = \{e^{-c(-\ln x)^r} : c > 0,\ r \neq 0\}$ of M;
- the subgroup $R = \{e^{-(-\ln x)^r} : r \neq 0\}$ of M;
- the element $\alpha(x) = 1 - x$ of M;
- the element $f(x) = e^{-\frac{1-x}{x}}$ of A;
- The set $F = \{\frac{a^x-1}{a-1} : a > 0,\ a \neq 1\}$

These groups and functions are closely connected. It was mentioned earlier that $N = \mathbb{R}^+ \times R$. An easy computation shows that $f\alpha f^{-1}(x) = e^{\frac{1}{\ln x}}$ so that $f\alpha f^{-1} \in R$, and in particular, f and $f\alpha$ determine the same right cosets of R and of N. That is, $Rf = Rf\alpha$ and $Nf = Nf\alpha$. This comes into play in the discussion below of the Dombi t-norms. Also, in some cases, α gives the duality between pairs of t-norms and t-conorms in a family even though α is not a member of the generating group or coset.

Now we list some families of t-norms (and t-conorms in some cases), and express their sets of generators as promised.

4 Families of strict t-norms

1. The Hamacher family (t-norms and generators):

$$\left\{\frac{xy}{x+y-xy+a(1-x-y+xy)} : a > 0\right\}$$

$$\left\{\frac{x}{x+a(1-x)} : a > 0\right\} = f^{-1}\mathbb{R}^+ f$$

Comments on this family: The group $\mathbb{R}^+$ gives only one t-norm, namely multiplication. However, the group $f^{-1}\mathbb{R}^+ f$, which is a conjugate of $\mathbb{R}^+$, gives a one-parameter family of t-norms. Since the group $f^{-1}\mathbb{R}^+ f$ contains only automorphisms, it does not generate any t-conorms.

2. The Jane Doe #2 family (t-norms and generators):

$$\left\{xye^{-a\ln x \ln y} : a > 0\right\}$$

$$\left\{\frac{1}{1-\ln x^a} : a > 0\right\} = f^{-1}\mathbb{R}^+$$

Comments on this family: The coset $f^{-1}\mathbb{R}^+$ gives a one-parameter family of t-norms. Since the coset contains only automorphisms, it does not generate any t-conorms.

3. The Jane Doe #3 family (t-norms and generators):

$$\left\{1-(1-(1-(1-x)^a)(1-(1-y)^a))^{\frac{1}{a}} : a > 0\right\}$$

$$\{1-(1-x)^a : a > 0\} = \alpha\mathbb{R}^+\alpha$$

Comments on this family: The group $\mathbb{R}^+$ gives only one t-norm, namely multiplication. However, the group $\alpha\mathbb{R}^+\alpha$, which is a conjugate of $\mathbb{R}^+$, gives a one-parameter family of t-norms. Since the group $\alpha\mathbb{R}^+\alpha$ contains only automorphisms, it does not generate any t-conorms.

4. The Schweizer family (t-norms and generators):

$$\left\{(x^{-a}+y^{-a}-1)^{-\frac{1}{a}} : a > 0\right\}$$

$$\left\{e^{-\left(\frac{1-x^a}{x^a}\right)} : a > 0\right\} = f\mathbb{R}^+$$

Comments on this family: The coset $f\mathbb{R}^+$ gives a one-parameter family of t-norms. Since the coset contains only automorphisms, it does not generate any t-conorms.

5. The Aczél-Alsina family (t-norms, t-conorms and generators):

$$\left\{e^{-((-\ln x)^r+(-\ln y)^r)^{\frac{1}{r}}} : r > 0\right\}$$

$$\left\{e^{-((-\ln x)^r+(-\ln y)^r)^{\frac{1}{r}}} : r < 0\right\}$$

$$\left\{e^{-(-\ln x)^r} : r \neq 0\right\} = R$$

Comments on this family: The only negation in this group of generators is the generator with $r = -1$, which gives the negation $e^{1/\ln x}$. This group gives a family of De Morgan systems, namely the t-norm with parameter r, $r > 0$, the negation $e^{1/\ln x}$, and the t-conorm with parameter $-r$.

6. The Jane Doe #1 family (t-norms, t-conorms and generators):

$$\left\{\frac{1}{1+\left[(\frac{1-x}{x})^r+(\frac{1-y}{y})^r+(\frac{1-x}{x})^r(\frac{1-y}{y})^r\right]^{\frac{1}{r}}}:r>0\right\}$$

$$\left\{\frac{1}{1+\left[(\frac{1-x}{x})^r+(\frac{1-y}{y})^r+(\frac{1-x}{x})^r(\frac{1-y}{y})^r\right]^{\frac{1}{r}}}:r<0\right\}$$

$$\left\{\frac{x^r}{x^r+(1-x)^r}:r\neq 0\right\}=f^{-1}Rf$$

Comments on this family: The only negation in this group of generators is the generator with $r=-1$, which gives the negation α. This group gives a family of De Morgan systems, namely the t-norm with parameter r, $r>0$, the negation $\alpha(x)=f^{-1}\left(e^{\frac{1}{\ln f(x)}}\right)=1-x$, and the t-conorm with parameter $-r$. The family of generators in this example is a conjugate of the group R of generators of the Aczél-Alsina family. Any conjugate of R will give a family of De Morgan systems having the same algebraic properties as those of the Aczél-Alsina family.

7. The Jane Doe #1-Hamacher family (t-norms, t-conorms and generators):

$$\left\{\frac{1}{1+[(\frac{1-x}{x})^r+(\frac{1-y}{y})^r+a((\frac{1-x}{x})^r(\frac{1-y}{y})^r]^{\frac{1}{r}}}:\begin{array}{c}a>0\\ r>0\end{array}\right\}$$

$$\left\{\frac{1}{1+[(\frac{1-x}{x})^r+(\frac{1-y}{y})^r+a((\frac{1-x}{x})^r(\frac{1-y}{y})^r]^{\frac{1}{r}}}:\begin{array}{c}a>0\\ r<0\end{array}\right\}$$

$$\left\{\frac{x^r}{x^r+a(1-x)^r}:a>0,\,r\neq 0\right\}=f^{-1}Nf$$

Comments on this family: The negations in this group of generators are those with parameters $a>0$, $r=-1$, which are the negations $(1-x)/(1+(a-1)x)$. This group gives a family of De Morgan systems, namely the t-norm with parameters $a>0$, $r>0$, the negation $(1-x)/(1+(a-1)x)$, and the t-conorm with parameters a, $-r$.

8. The Dombi family (t-norms, t-conorms and generators):

$$\left\{\frac{1}{(1+(\frac{1-x}{x})^r+(\frac{1-y}{y})^r)^{\frac{1}{r}}}:r>0\right\}$$

$$\left\{\frac{1}{(1+(\frac{1-x}{x})^r+(\frac{1-y}{y})^r)^{\frac{1}{r}}}:r<0\right\}$$

$$\left\{e^{-(\frac{1-x}{x})^r}:r\neq 0\right\}=Rf$$

Comments on this family: The coset Rf has no negation in it. The t-norm with parameter r ($r>0$), is dual to the t-conorm with parameter r ($r<0$), with respect to the negation α. Denote the element $e^{-(\frac{1-x}{x})^r}$ of Rf by t_rf. Then the element $f\alpha f^{-1}=e^{1/\ln x}=t_{-1}\in R$. For a generator t_rf of a t-norm in the Dombi family, $t_rf\alpha=t_rf\alpha f^{-1}f=t_rt_{-1}f\in Rf$.

9. The Frank family (t-norms and generators):

$$\left\{\log_a\left(1+\frac{(a^x-1)(a^y-1)}{a-1}\right):a>0,\ a\neq 1\right\}$$

$$\left\{\frac{a^x-1}{a-1}:a>0,\ a\neq 1\right\}=F$$

Comments on this family: The set of Frank t-norms is the set of t-norms $\triangle$ satisfying the equation $x\triangle y+x\triangledown y=x+y$, where $\triangledown$ is the t-conorm dual to $\triangle$ with respect to the negation α. This set of t-norms does not fit the pattern of the other well-known families of strict t-norms, as it does not come from a group or a coset of a group of generators. Nevertheless, the negation α plays a critical role in its defining equation. The dual t-conorms for the Frank family relative to α and their generators are

$$\left\{1-\log_a\left(1+\frac{(a^{1-x}-1)(a^{1-y}-1)}{a-1}\right):\begin{array}{c}a>0\\ a\neq 1\end{array}\right\}$$

$$\left\{\frac{a^{1-x}-1}{a-1}:a>0,\ a\neq 1\right\}=F\alpha$$

This family can be reparametrized as follows: For each $a>1$, $a\neq 1$, write $a=e^{-b}$. Then for $b>0$, $e^{-b}>0$ and $e^{-b}\neq 1$. So replacing a by e^{-b} reparametrizes this family with the parameters the positive reals. This makes the parameters into a group under multiplication. However, this operation is not related to the group operation of composition of the generators. There does not appear to be a group theoretic representation of this family, but this remains an open question.

5 Families of nilpotent t-norms

1. The Schweizer-Sklar family (t-norms and L-generators):

$$\left\{((x^a + y^a - 1) \vee 0)^{\frac{1}{a}} : a > 0\right\}$$

$$\{x^a : a > 0\} = \mathbb{R}^+$$

Comments on this family: The set of generators of this family forms a group, inducing a group structure on the set of Schweizer-Sklar t-norms. The identity of this group is the Łukasiewicz t-norm $(x + y - 1) \vee 0$, corresponding to the parameter $a = 1$.

2. The Yager family (t-norms and L-generators):

$$\left\{(1 - ((1-x)^a + (1-y)^a)^{\frac{1}{a}}) \vee 0 : a > 0\right\}$$

$$\{1 - (1-x)^a : a > 0\} = \alpha^{-1}\mathbb{R}^+\alpha$$

Comments on this family: The set $\alpha^{-1}\mathbb{R}^+\alpha$ of generators forms a group, being a conjugate of $\mathbb{R}^+$. The Schweizer-Sklar and Yager families have only the Lukasiewicz t-norm in common, corresponding to the fact that $\mathbb{R}^+ \cap \alpha^{-1}\mathbb{R}^+\alpha = \{1\}$.

3. The Weber family (t-norms and L-generators):

$$\{(a(x + y - 1) - (a-1)xy) \vee 0 : a > 0\}$$

$$\left\{1 - \log_a(1 - (1-a)(1-x)) : \begin{matrix} a > 0 \\ a \neq 1 \end{matrix}\right\} = \alpha^{-1}F^{-1}\alpha$$

Comments on this family: This set of generators does not form a group and is not a coset of a subgroup of M. The generators of this family, and the next two, are related to the generators of the Frank family of strict t-norms or t-conorms.

4. Jane Doe #4 (t-norms and L-generators):

$$\left\{\left(\tfrac{1}{a}\left(x + y - 1 + (a-1)xy\right)\right) \vee 0 : a > 0,\ a \neq 1\right\}$$

$$\{\log_a(1 + (a-1)x) : a > 0,\ a \neq 1\} = F^{-1}$$

5. Jane Doe #6 (t-norms and L-generators):

$$\left\{\log_a\left(1 + \sigma_a\left(\left(\sigma_a\left(a^x + a^y - a - 1\right)\right) \vee 0\right)\right) : a > 0, a \neq 1\right\} \text{ where } \sigma_a = \tfrac{|a-1|}{a-1}$$

$$\left\{\frac{a^x - 1}{a - 1} : a > 0, a \neq 1\right\} = F$$

Just why the generators of so many of the well known families of strict and nilpotent t-norms are simply group theoretic combinations—for example, cosets or conjugates of the groups $\mathbb{R}^+$, R, and N—is not too clear. As mentioned, the Frank family (and consequently the Weber, Jane Doe #4, and Jane Doe #6 nilpotent families with related generating sets) do not fit that mold. It is of interest to note that the groups R and N come from $\mathbb{R}^+$ rather directly, N being the normalizer of $\mathbb{R}^+$ and R being a complementary factor to $\mathbb{R}^+$ in N, as indicated in Section 3.

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