Algebraic Aspects of Fuzzy Sets and Fuzzy Logic

M. Gehrke, C. Walker, and E. Walker
Department of Mathematical Sciences
New Mexico State University
Las Cruces, New Mexico 88003-0001, USA

1 Introduction

This paper is expository. It is mainly a survey of some of our work on the algebraic systems that arise in fuzzy set theory and logic. We include some of the proofs here and refer to the papers [18, 19, 20, 21, 22] for further details. Our point of view is the algebraic one: when are the various systems that arise isomorphic, and what are their symmetries (automorphisms)? The bulk of the material centers around t-norms, and a typical concern is with the unit interval endowed with its natural order structure, a t-norm, and a negation. A fundamental problem is to determine when two such algebraic systems are isomorphic.

Section 2.1 introduces the notions involved in the structure of a deMorgan system on the unit interval. Section 2.1 gives some basic facts about the unit interval $\mathbb{I} = ([0,1], \leq)$ with its order structure. This includes some information about various groups that will play a role throughout. Section 2.2 introduces t-norms, gives representation theorems for them, and determines their isomorphy, and their automorphism groups. Section 2.3 does for negations what Section 2.2 does for t-norms. In Section 2.4, deMorgan systems are introduced. These concern the unit interval with its usual order, a t-norm, a negation, and the corresponding t-conorm.

In Section 3, isomorphisms of deMorgan systems with strict t-norms are discussed. A typical result is that any deMorgan system with strict t-norm and strong negation is isomorphic to one whose t-norm is multiplication. The non-uniqueness of the negation in a strict deMorgan system is discussed in Section 3.1. The generators of strict t-norms are determined up to elements of the multiplicative group of positive real numbers. This group can be viewed as a subgroup of the automorphism group of the unit interval, and its normalizer in that group yields a special set of t-norms. In Section 3.2, we determine that normalizer and give explicit formulas for the resulting t-norms, t-conorms and negations.

In Section 4, isomorphisms of various systems with nilpotent t-norms are discussed. A typical result is that any deMorgan system with nilpotent t-norm and strong negation is isomorphic to one whose t-norm is the well-known Łukaciewisz t-norm. This provides a unique representation of nilpotent t-norms by elements of the automorphism group of $\mathbb{I}$, which puts a (nearly) natural group structure on the set of nilpotent t-norms and also leads to theorems similar to those for strict t-norms. In Section 4.1, we explore general relationships among negations, convex Archimedean nilpotent t-norms, and automorphisms of the unit interval $\mathbb{I}$. Each nilpotent t-norm has a (strong) negation naturally associated with it. The notions of Stone and Boolean systems are introduced. A deMorgan system is shown to be a Boolean system if and only if the t-norm is nilpotent and the negation is the one naturally associated with it.

In Section 5 we consider averaging operators—binary operations $\dot{+}$ on the unit interval that are commutative; strictly increasing in each variable; convex (continuous); idempotent; and bisymmetric. All averaging operators are isomorphic to the arithmetic mean via an automorphism f of the unit interval (a generator for the averaging operator) that takes the given average of two elements x and y to the arithmetic mean of $f(x)$ and $f(y)$. Averaging operators provide a (continuous) scaling of the unit interval that is not provided by the lattice structure. We consider mean systems $(\mathbb{I}, \dot{+})$, where $\dot{+}$ is an averaging operator on $\mathbb{I}$ and note that these algebras have no nontrivial automorphisms. In Section 5.1 we show that each averaging operator $\dot{+}$ on the unit interval naturally defines a negation η by the equation $x \dot{+} \eta(x) = 0 \dot{+} 1$ and the averaging operator is "self-dual" with respect to this negation. In Section 5.2 we relate an averaging operator to the nilpotent t-norms that determine the same negation and find a natural one-to-one correspondence between averaging operators and nilpotent t-norms. Corresponding averaging operators and nilpotent t-norms determining the same negation. This correspondence relates the Łukaciewisz t-norm to the arithmetic mean, both of which lead to the standard negation $1 - x$, for example. We consider what happens in the general case. In Section 5.3 we consider deMorgan systems with averaging operators and generalize the families of Frank t-norms and nearly Frank t-norms in this setting.

In Section 6 we develop the basic theory of t-norms, negations, and t-conorms on interval-valued fuzzy sets, where the unit interval is replaced by the lattice $\mathbb{I}^{[2]} = \left([0,1]^{[2]}, \leq\right)$ with $[0,1]^{[2]} = \{(a,b) : a, b \in [0,1], a \leq b\}$. There is a view that models based on the unit interval are inadequate, that assigning an exact number to an expert's opinion is too restrictive, and that the assignment of an interval of values is more realistic. The basic theory goes through, and Section 6.2 develops that theory. We also look at deMorgan, weak Boolean and Stone systems on the lattice of subintervals and compare properties of related systems with those on the unit interval.

In Section 7 the setup of a mathematical propositional logic is given in algebraic terms, describing exactly when two choices of truth value algebras give the same logic. The propositional logic obtained when the algebra of truth values is the real numbers in the unit interval equipped with minimum, maximum and $\neg x = 1 - x$ for conjunction, disjunction and negation, respectively, is the standard propositional fuzzy logic. This is shown to be the same as three-valued logic. The propositional logic obtained when the algebra of truth values is the set $\{(a, b) \mid a \leq b \text{ and } a, b \in [0, 1]\}$ of subintervals of the unit interval with component-wise operations, is propositional interval-valued fuzzy logic. This is shown to be the same as the logic given by a certain four element lattice of truth values. Since both of these logics are equivalent to ones given by finite algebras, it follows that there are finite algorithms for determining when two statements are logically equivalent within either of these logics. On this topic, normal forms are described for both of these logics.

2 The unit interval, t-norms, t-conorms and negations

A **fuzzy subset** A of a set S is a mapping $A : S \to [0, 1]$. Operations on the set of all such fuzzy subsets of S come from operations on $[0, 1]$. Standard ones are $\wedge, \vee$, and $'$ given by

$$\begin{aligned} (A \wedge B)(s) &= \min\{A(s), B(s)\} \\ (A \vee B)(s) &= \max\{A(s), B(s)\} \\ A'(s) &= 1 - A(s) \end{aligned}$$

Viewing subsets of S as mappings $S \to \{0, 1\}$, these operations clearly generalize the usual notions of intersection, union, and complement. But there are many other such generalizations and a huge literature dealing with them. There are extensive bibliographies in [6], [12], and [26] for example. Our main concern will be with special classes of those connectives on the unit interval, particularly strict and nilpotent Archimedean t-norms and t-conorms, and strong negations. We give the notion of generators a new emphasis, and investigate systematically the isomorphisms between deMorgan systems on the unit interval.

2.1 The unit interval

The interval $[0, 1]$ is endowed with an order $\leq$, and has many other mathematical features. But the system $\mathbb{I} = ([0, 1], \leq)$ is the basic building block of the theory. Our concern will be with this system with additional operations on it, namely t-norms and the like. We begin with some discussion of automorphisms and antiautomorphisms of $\mathbb{I}$.

Definition 1 *An **automorphism** of $\mathbb{I}$ is a one-to-one mapping f of $\mathbb{I}$ onto $\mathbb{I}$ such that $f(a) \leq f(b)$ if and only if $a \leq b$. An **antiautomorphism** of $\mathbb{I}$ is a one-to-one mapping g of $\mathbb{I}$ onto $\mathbb{I}$ such that $g(a) \geq g(b)$ if and only if $a \leq b$.*

Thus automorphisms preserve $\leq$ and antiautomorphisms reverse $\leq$. It should be noted that $f(0) = 0$ and $f(1) = 1$ for automorphisms f and $g(0) = 1$ and $g(1) = 0$ for antiautomorphisms g. Since discontinuities of monotone functions are jumps, these automorphisms and antiautomorphisms are continuous. Of course, automorphisms are strictly increasing, and antiautomorphisms are strictly decreasing. These mappings are plentiful. Any continuous strictly increasing map connecting $(0, 0)$ and $(1, 1)$ in the plane is an automorphism of $\mathbb{I}$.

Let $Map(\mathbb{I})$ be the set consisting of all automorphisms and all antiautomorphisms of $\mathbb{I}$, and let $Aut(\mathbb{I})$ be the set of all automorphisms of $\mathbb{I}$. The elements of $Map(\mathbb{I})$ are functions, and may be composed. That is, if f and g are in $Map(\mathbb{I})$, fg is the element of $Map(\mathbb{I})$ given by $(fg)(x) = f(g(x))$. With this operation, $Map(\mathbb{I})$ is a **group.** That is, composition of functions is a binary operation on $Map(\mathbb{I})$ that is associative, has an identity, and every element has an inverse. This means that

- $f(gh) = (fg)h$.
- There is an element id in $Map(\mathbb{I})$ such that $\text{id} \circ f = f \circ \text{id} = f$ for all f. (The function id is the function given by $\text{id}(x) = x$ for all x. It is called the **identity** of the group.)
- For each $f \in Map(\mathbb{I})$, there is an element $f^{-1} \in Map(\mathbb{I})$ such that $f \circ f^{-1} = f^{-1} \circ f = \text{id}$. (The element f^{-1} is simply the inverse of f as a function on $[0, 1]$.)

$Aut(\mathbb{I})$ is a **subgroup** of $Map(\mathbb{I})$. This means that the restriction of the operation on $Map(\mathbb{I})$ to $Aut(\mathbb{I})$ makes $Aut(\mathbb{I})$ into a group. This subgroup happens to be **normal**, that is, for every element f of $Aut(\mathbb{I})$ and g of $Map(\mathbb{I})$, the element $g^{-1}fg$ belongs to the subgroup $Aut(\mathbb{I})$. Elements of the form $g^{-1}fg$ are **conjugates** of f.

There are a couple of other important subgroups.

- Each positive real number r gives an automorphism of $\mathbb{I}$ by $r(x) = x^r$. Identifying r with this automorphism, the set $\mathbb{R}^+$ of positive real numbers is a subgroup of $Aut(\mathbb{I})$.
- For any subset S of $Map(\mathbb{I})$ the set $\{x \in Map(\mathbb{I}) : xs = sx \text{ for all } s \in S\}$ is the **centralizer** $Z(S)$ of S in $Map(\mathbb{I})$ and is a subgroup of $Map(\mathbb{I})$.

A particularly important antiautomorphism is α, given by $\alpha(x) = 1 - x$. Its centralizer consists of those $f \in Map(\mathbb{I})$ such that $f\alpha = \alpha f$. In this particular case, we will only be interested in those f which are in $Aut(\mathbb{I})$, that is, in the centralizer of α in $Aut(\mathbb{I})$, which is the group

$$Z(\{\alpha\}) \cap Aut(\mathbb{I}) = \{f \in Aut(\mathbb{I}) : f\alpha = \alpha f\}$$

For ease of notation, we are going to denote this group by $Z(\alpha)$, and more generally, for any $g \in Map(\mathbb{I})$,

$$Z(g) = \{f \in Aut(\mathbb{I}) : fg = gf\}$$

The group $Z(\alpha)$ consists exactly of those elements of $Aut(\mathbb{I})$ which commute with α, which is equivalent to

$$\begin{aligned} f\alpha(x) &= f(1-x) \\ &= \alpha f(x) \\ &= 1 - f(x) \end{aligned}$$

or that

$$f(x) + f(1-x) = 1.$$

Elements of $Z(\alpha)$ are easy to construct. An easy computation establishes the following.

Theorem 2 $Z(\alpha) = \{\frac{\alpha f \alpha + f}{2} : f \in Aut(\mathbb{I})\}$.

The map $\Phi : Aut(\mathbb{I}) \to Z(\alpha) : f \to \frac{\alpha f \alpha + f}{2}$ fixes $Z(\alpha)$ elementwise, but its group theoretical significance is not clear. For example, it is not a homomorphism: $\Phi(f)\Phi(g) = \Phi(f\Phi(g)) \neq \Phi(fg)$. Perhaps the pertinent algebraic structure here is the monoid $\mathbb{S} = (Aut\,(I)\,, *)$ where $f * g = f \circ \Phi(g)$. The group $Z\,(\alpha)$ is the unique maximal subgroup of $\mathbb{S}$, and in this context $\Phi : \mathbb{S} \to Z(\alpha)$ is a retract.

We will need the following later on. First, notice that for any $f \in Map(\mathbb{I})$ and any subgroup G of $Map(\mathbb{I})$,

$$f^{-1}Gf = \{f^{-1}gf : g \in G\}$$

is a subgroup of $Map(\mathbb{I})$.

Proposition 3 *For any f and $g \in Map(\mathbb{I})$,*

$$\left(f^{-1}\mathbb{R}^{+}f\right) \cap \left(g^{-1}Z(\alpha)g\right) = \{1\}$$

Proof. If $f^{-1}rf = g^{-1}zg$, then $gf^{-1}r = zgf^{-1}$. There is $x \in [0,1]$ such that $gf^{-1}(x) = \frac{1}{2}$. For this x, $gf^{-1}r(x) = zgf^{-1}(x) = z\left(\frac{1}{2}\right) = \frac{1}{2}$, and so $gf^{-1}(x^r) = \frac{1}{2}$. But $gf^{-1}(x) = \frac{1}{2}$, and since gf^{-1} is one-to-one, $r = 1$ and the proposition follows. ■

2.2 Convex Archimedean t-norms

We will put additional structure on the system $\mathbb{I}$, and first we consider t-norms. They are generalizations of intersection and are one of the fundamental objects of interest in fuzzy set theory and logic.

Definition 4 *A **t-norm** is a binary operation $\circ$ on $[0,1]$ such that for all $x, y, z \in [0,1]$,*

1. $1 \circ x = x$
2. $x \circ y = y \circ x$
3. $(x \circ y) \circ z = x \circ (y \circ z)$
4. *The operation* $\circ$ *is increasing in each variable. That is,* $x \leq x_1$ *and* $y \leq y_1$ *imply that* $x \circ y \leq x_1 \circ y_1$.

Thus a binary operation on $[0, 1]$ is a t-norm if 1 is an identity, it is commutative, associative, and increasing in each variable. Of course, the associative property 3 gives unambiguous meaning to $x_1 \circ x_2 \circ ... \circ x_n$, and in particular to $x \circ x \circ ... \circ x$, which we write as x^n, where n is the number of x's. We have to be a little careful with this: x is a real number so for any real number r, x^r has meaning as the r-th power of x. The context will make clear the meaning of x^n. A t-norm $\circ$ has the following additional properties.

- $0 \circ x = 0$. (This follows since $0 \circ x \leq 0 \circ 1 = 0$.)
- $x \circ y = 1$ if and only if $x = y = 1$. (Clearly $1 = 1 \circ 1$. If $x \circ y = 1$, then $y = 1 \circ y \geq x \circ y = 1$ and similarly $x = 1$.)

Here are a few examples of t-norms

- minimum:
$$x \circ y = x \wedge y$$
- multiplication:
$$x \circ y = x \cdot y$$
- Łukaciewisz t-norm:
$$x \blacktriangle y = (x + y - 1) \vee 0$$
- Hamacher t-norms:
$$x \bigtriangleup_H y = \frac{xy}{a + (1 - a)(x + y - xy)} \text{ for } a \geq 0$$
- Frank t-norms:
$$x \bigtriangleup_F y = \log_a \left[1 + \frac{(a^x - 1)(a^y - 1)}{a - 1}\right] \text{ for } a > 0,\ a \neq 1$$
- Yager t-norms:
$$x \bigtriangleup_Y y = \left(1 - \left(((1 - x)^a + (1 - y)^a)^{\frac{1}{a}}\right)\right) \vee 0 \text{ for } a \geq 1$$

- Richman t-norms:

$$x \triangle_R y = e^{-((-\ln x)^a + (-\ln y)^a)^{\frac{1}{a}}} \text{ for } a > 0$$

- generalized Łukaciewisz t-norms:

$$x \blacktriangle_a y = ((x^a + y^a - 1) \vee 0)^{\frac{1}{a}} \text{ for } a \neq 0$$

Let $\circ$ be a t-norm and consider the system $(\mathbb{I}, \circ)$. This system is simply $\mathbb{I}$ with an additional structure on it, namely the operation $\circ$. Let $\diamond$ be another t-norm on $\mathbb{I}$. We need to make precise the notion of the systems $(\mathbb{I}, \circ)$ and $(\mathbb{I}, \diamond)$ being structurally the same. One of our primary concerns is with such questions.

Definition 5 *Let $\circ$ and $\diamond$ be t-norms. The systems $(\mathbb{I}, \circ)$ and $(\mathbb{I}, \diamond)$ are **isomorphic** if there is an element $h \in Aut(\mathbb{I})$ such that $h(x \circ y) = h(x) \diamond h(y)$. We write $(\mathbb{I}, \circ) \approx (\mathbb{I}, \diamond)$. The mapping h is an **isomorphism**. The t-norms $\circ$ and $\diamond$ are **isomorphic** if $(\mathbb{I}, \circ) \approx (\mathbb{I}, \diamond)$.*

This means that the systems $([0,1], \leq, \circ)$ and $([0,1], \leq, \diamond)$ are isomorphic in the sense of universal algebra: there is a one-to-one map from $[0,1]$ onto $[0,1]$ that preserves the operations and relations involved.

Isomorphism of t-norms is an equivalence relation and partitions t-norms into equivalence classes. The t-norm min is rather special. A t-norm $\circ$ is **idempotent** if $a \circ a = a$ for all $a \in [0,1]$. If $\circ$ is idempotent, then for $a \leq b$, $a = a \circ a \leq a \circ b \leq a \circ 1 = a$, so $\circ = \min$. Thus min is the *only* idempotent t-norm. It is in an equivalence class all by itself.

An isomorphism of a system with itself is called an **automorphism**. It is easy to show that the set of automorphisms of $(\mathbb{I}, \circ)$ is a subgroup of $Aut(\mathbb{I})$. Thus, with each t-norm $\circ$, there is a group associated with it, namely its **automorphism group**

$$Aut(\mathbb{I}, \circ) = \{f \in Aut(\mathbb{I}) : f(x \circ y) = f(x) \circ f(y)\}$$

For the t-norm $a \wedge b = \min\{a, b\}$, it is clear that $Aut(\mathbb{I}, \wedge) = Aut(\mathbb{I})$. For the t-norm multiplication,

$$\begin{aligned} Aut(\mathbb{I}, \cdot) &= \{f \in Aut(\mathbb{I}) : f(xy) = f(x)f(y) \text{ for all } x, y \in [0,1]\} \\ &= \left\{f \in Aut(\mathbb{I}) : f(x) = x^c, c \in \mathbb{R}^+\right\} \approx \mathbb{R}^+. \end{aligned}$$

This is a well-known, classical result related to one of the four basic functional equations of Cauchy (see, for example, [4]).

If H is a subgroup of a group G, and $g \in G$, then we have already noted that $g^{-1}Hg = \{g^{-1}hg : h \in H\}$ is a subgroup of G. This subgroup is said to be **conjugate to** H, or a **conjugate of** H. The map $h \to g^{-1}hg$ is an isomorphism from H to its conjugate $g^{-1}Hg$. By an **isomorphism** from a group G to a group H we mean a one-to-one onto map $\varphi : G \to H$ such that $\varphi(xy) = \varphi(x)\varphi(y)$.

Theorem 6 *If two t-norms are isomorphic then their automorphism groups are conjugate.*

Proof. Suppose that $\circ$ and $\diamond$ are isomorphic. Then there is an isomorphism $f : (\mathbb{I}, \circ) \to (\mathbb{I}, \diamond)$. The map $g \to f^{-1}gf$ is an isomorphism from $Aut(\mathbb{I}, \diamond)$ to $Aut(\mathbb{I}, \circ)$, so $f^{-1}Aut(\mathbb{I}, \diamond)f = Aut(\mathbb{I}, \circ)$. ■

Definition 7 *A t-norm $\circ$ is **convex** if whenever $x \circ y \leq c \leq x_1 \circ y_1$, then there is an r between x and x_1 and an s between y and y_1 such that $c = r \circ s$. A t-norm $\circ$ is **Archimedean** if for each $a, b \in (0, 1)$, there is a positive integer n (where n depends on a and b) such that*

$$a^n = \overbrace{a \circ a \circ \cdots \circ a}^{n \text{ times}} < b$$

For t-norms, the condition of convexity is equivalent to continuity in the usual topology on the unit interval. We refer to the condition as convex. This formulation has the advantage of being strictly order theoretic, allowing us to remain within the algebraic context of $\mathbb{I}$ as a lattice. For convex t-norms, the condition for Archimedean simplifies, as the following well-known proposition attests.

Proposition 8 *The following are equivalent for a convex t-norm $\circ$.*

1. *$\circ$ is Archimedean.*
2. *$a \circ a < a$ for all $a \in (0, 1)$.*

The theorem below is fundamental in determining equivalences of convex Archimedean t-norms. It has usually been thought of as a theorem about representing t-norms by generators. A principle reference is [29]. The theorem in essence goes back at least to Abel [1]. There is a proof for the strict case in [3], some discussion in [39], and a proof in [40]. We will give only a very brief outline of a proof here.

Theorem 9 *If $\circ$ is a convex Archimedean t-norm then there is an $a \in [0, 1)$ and an isomorphism*

$$f : \mathbb{I} \to ([a, 1], \leq)$$

such that

$$f(x \circ y) = \max\{f(x)f(y), a\}$$

for all $x, y \in [0, 1]$. Also if $g : \mathbb{I} \to ([b, 1], \leq)$ is another such isomorphism, then $g(x \circ y) = \max\{g(x)g(y), b\}$ if and only if $f = rg$ for some $r > 0$.

Proof. The proof consists of a construction of a map f satisfying $f(x \circ y) = f(x) \cdot f(y)$ for $x \circ y \neq 0$. An increasing sequence $\{x_n\}_{n=-\infty}^{\infty}$ in $[0, 1)$ is defined inductively by the condition $x_n \circ x_n = x_{n-1}$. The set of all points of the form $x_{i_1} \circ x_{i_2} \circ \cdots \circ x_{i_n}$ is dense in the unit interval. Define a function f on the sequence $\{x_n : x_n \neq 0\}$ by

$$f(x_n) = x_0^{2^{-n}} \text{ if } x_n \neq 0$$

The function f can be extended to finite nonzero products under $\circ$ of the elements of this sequence by

$$f\left(x_{i_1} \circ x_{i_2} \circ \cdots \circ x_{i_n}\right) = f(x_{i_1}) f(x_{i_2}) \cdots f(x_{i_n}).$$

and then extended to an isomorphism from $\mathbb{I}$ to $([f(0), 1], \leq)$ by convexity. Then $f(x \circ y) = f(x)f(y)$ whenever $x \circ y \neq 0$, so that $x \circ y = f^{-1}(f(x)f(y))$ if $f(x)f(y) \geq f(0)$ and $x \circ y = 0$ otherwise.

Suppose that an isomorphism $g : I \to ([g(0), 1], \leq)$ gives the same t-norm as the f just constructed. Take $r = (\ln x_0) / (\ln g(x_0))$. Then $rg(x_0) = (g(x_0))^r = x_0 = f(x_0)$, and it can be seen from the construction of f that rg must agree with f on the points x_n and hence everywhere. Conversely, it is easy to show that if $f = rg$, then f and g give the same t-norm. ■

A function f such that $x \circ y = f^{-1}(\max\{f(x)f(y), f(0)\})$ is called a **generator** of the t-norm $\circ$. Two functions f and g are generators of the same t-norm if and only if $f = rg$ for some $r > 0$, that is, $f(x) = (g(x))^r$ for all $x \in [0, 1]$.

A convex, Archimedean t-norm $\circ$ is **nilpotent** if for $a \neq 1$, $a^n = 0$ for some positive integer n, the n depending on a. It is clear from the theorem that $\circ$ is nilpotent if and only if $f(0) > 0$. So convex Archimedean t-norms fall naturally into two classes: nilpotent ones and those not nilpotent. Those not nilpotent are called **strict.** In the next section, we start by examining the strict ones.

Historically, Archimedean t-norms have been represented by maps $g : I \to [0, \infty]$, where g is a strictly decreasing function with $0 < g(0) \leq \infty$ and $g(1) = 0$. In this case the binary operation satisfies

$$g(x \circ_{g^+} y) = \min\{g(x) + g(y), g(0)\}$$

and since this minimum is in the range of g,

$$x \circ_{g^+} y = g^{-1}(\min\{g(x) + g(y), g(0)\})$$

These two types of representations give the same t-norms. In particular, if $g : I \to [0, \infty]$ is a continuous, strictly decreasing function, with $0 < g(0) \leq \infty$ and $g(1) = 0$, let $f(x) = e^{-g(x)}$, $x \circ_f y = f^{-1}(\max\{f(x)f(y), f(0)\})$ and $x \circ_g y = g^{-1}(\min\{g(x) + g(y), g(0)\})$. Then $f : \mathbb{I} \to [f(0), 1]$ is an isomorphism and $\circ_{g^+} = \circ_f$. (See [40], for example.) We use the multiplicative representation, since this allows us to remain within the context of the unit interval and, in the case of strict t-norms, to have a natural group structure on the set of generators. (For nilpotent t-norms we will develop an alternate representation that allow us to have a natural group structure not only on the set of generators, but on the set of t-norms as well.)

2.2.1 Strict t-norms

We restate the representation theorem for the strict t-norm case.

Theorem 10 *An Archimedean t-norm* $\circ$ *is strict if and only if there is an element* $f \in Aut(\mathbb{I})$ *such that* $f(x \circ y) = f(x)f(y)$*. Another element* $g \in Aut(\mathbb{I})$ *satisfies this condition if and only if* $f = rg$ *for some* $r > 0$*.*

So a generator of a strict t-norm $\circ$ is just an isomorphism from $Aut(\mathbb{I}, \circ)$ to $Aut(\mathbb{I}, \cdot)$. This means that a t-norm is strict if and only if it is isomorphic to multiplication.

Corollary 11 *For any strict t-norm* $\circ$*,* $Aut(\mathbb{I}, \circ) \approx Aut(\mathbb{I}, \cdot)$*.*

Corollary 12 *For any two strict t-norms* $\circ$ *and* $\diamond$*,* $Aut(\mathbb{I}, \circ) \approx Aut(\mathbb{I}, \diamond)$*.*

Additional properties of strict t-norms $\circ$ are these:

- On $(0, 1)$ the operation $\circ$ is strictly increasing in each variable. In fact, if $\circ$ is strict, then $f(x) = y \circ x$ is a one-to-one map of $[0, 1]$ onto $[0, y]$. This follows from the convexity and strict monotonicity of $\circ$.

- If $\circ$ is strict, then $f(x) = x \circ x$ is an automorphism and $g(x) = (1 - x) \circ (1 - x)$ is an antiautomorphism of $\mathbb{I}$. Again, this follows from the convexity and strict monotonicity of $\circ$.

Among the examples of t-norms listed earlier, the strict t-norms are multiplication, the Hamacher t-norms, the Frank t-norms, the Richman t-norms, and the generalized Łukaciewisz t-norms with $a < 0$.

Call two automorphisms f and g **equivalent** if they give the same strict t-norm, and write $f \sim g$. Then $\sim$ is an equivalence relation and so induces a partition of the group $Aut(\mathbb{I})$. The members of this partition are the **right cosets** $\{\mathbb{R}^+ f : f \in Aut(\mathbb{I})\}$ of $\mathbb{R}^+$. So the set of strict t-norms of $\mathbb{I}$ is in natural one-to-one correspondence with the right cosets in $Aut(\mathbb{I})$ of the subgroup $\mathbb{R}^+$. Rephrasing, we have

Corollary 13 *For an automorphism* f *of* $\mathbb{I}$*, let* $\circ_f$ *be given by* $x \circ_f y = f^{-1}(f(x) f(y))$*. Then*

$$\circ_f \to \mathbb{R}^+ f$$

is a one-to-one correspondence between the strict t-norms on $[0, 1]$ *and the right cosets of the subgroup* $\mathbb{R}^+$ *in* $Aut(\mathbb{I})$*.*

Let $a \in (0, 1)$. The set $G_a = \{f \in Aut(\mathbb{I}) : f(a) = a\}$ is a subgroup of $Aut(\mathbb{I})$ with exactly one element in each right coset $\mathbb{R}^+ f$. Thus for $f \in G_a$, the map $\circ_f \mapsto f$ is a one-to-one correspondence between the strict t-norms on $\mathbb{I}$ and the group G_a.

We know that for strict t-norms $\circ$ and $\diamond$, the systems $(\mathbb{I}, \circ)$ and $(\mathbb{I}, \diamond)$ are isomorphic. We spell out exactly what those isomorphisms are.

Theorem 14 *Let $\circ$ and $\diamond$ be strict t-norms with generators f and g, respectively. Then $h : (\mathbb{I}, \circ) \to (\mathbb{I}, \diamond)$ is an isomorphism if and only if $g^{-1}rf = h$ for some $r > 0$. That is, the set of isomorphisms from $(\mathbb{I}, \circ)$ to $(\mathbb{I}, \diamond)$ is the set*

$$g^{-1}\mathbb{R}^+f = \{g^{-1}rf : r \in \mathbb{R}^+\}.$$

Proof. An isomorphism $h : (\mathbb{I}, \circ) \to (\mathbb{I}, \diamond)$ gets an isomorphism $gh : (\mathbb{I}, \circ) \to (\mathbb{I}, \cdot)$ which must be rf for some $r \in \mathbb{R}^+$. So $h = g^{-1}rf$. For any r, $g^{-1}rf$ is an isomorphism. ■

Corollary 15 *Let f be a generator of the strict t-norm $\circ$. Then*

$$Aut(\mathbb{I}, \circ) = f^{-1}\mathbb{R}^+f \approx \mathbb{R}^+.$$

Proof. The set of automorphisms of $(\mathbb{I}, \circ)$ is $f^{-1}\mathbb{R}^+f$. It is a subgroup of $Aut(\mathbb{I})$, and is isomorphic to $\mathbb{R}^+$ via the mapping $f^{-1}rf \to r$. ■

Corollary 16 $Aut(\mathbb{I}, \cdot) = \mathbb{R}^+$.

2.2.2 Nilpotent t-norms

There are two basic facts about nilpotent (convex Archimedean) t-norms: any two are isomorphic, and each has a trivial automorphism group. This leads to an alternate form of the representation theorem for nilpotent t-norms for which the generating function is a (unique) automorphism of $\mathbb{I}$.

Theorem 17 *Let $\circ$ and $\diamond$ be nilpotent t-norms with generators f and g respectively. Then $\circ$ and $\diamond$ are isomorphic and $g^{-1}rf$ is the unique isomorphism from $(\mathbb{I}, \circ)$ to $(\mathbb{I}, \diamond)$, where $r \in \mathbb{R}^+$ with $g(0) = (f(0))^r$.*

Proof. We may take $r = 1$, so that $f(0) = g(0)$. First note that $g^{-1}f \in Aut(\mathbb{I})$. We need to show that $g^{-1}f(a \circ b) = g^{-1}f(a) \diamond g^{-1}f(b)$, that is, that

$$g^{-1}ff^{-1}(\max\{f(a)f(b), f(0)\}) = g^{-1}(\max\{gg^{-1}f(a)gg^{-1}f(b), g(0)\})$$

which is clear since $f(0) = g(0)$. Suppose that $\varphi : (\mathbb{I}, \circ) \to (\mathbb{I}, \diamond)$ is an isomorphism. Then

$$\varphi f^{-1}(\max\{f(a)f(b), f(0)\}) = g^{-1}(\max\{(g\varphi)(a)(g\varphi)(b), g\varphi(0)\})$$

Thus

$$f^{-1}(\max\{f(a)f(b), f(0)\}) = \varphi^{-1}g^{-1}(\max\{(g\varphi)(a)(g\varphi)(b), g\varphi(0)\})$$

Since f and $g\varphi$ generate the same nilpotent t-norm and agree on 0, $f = g\varphi$ or $\varphi = g^{-1}f$ as asserted. ■

Corollary 18 *If $\circ$ is a nilpotent t-norm, then $Aut(\mathbb{I}, \circ) = \{\text{id}\}$.*

Corollary 19 *If $\circ$ is a nilpotent t-norm, then $(\mathbb{I}, \circ) \approx (\mathbb{I}, \blacktriangle)$ where $\blacktriangle$ is the Łukaciewisz t-norm*

$$x \blacktriangle y = (x + y - 1) \vee 0$$

Here is the alternate representation theorem for the nilpotent t-norm case.

Theorem 20 *An Archimedean t-norm $\circ$ is nilpotent if and only if there is an element $f \in Aut(\mathbb{I})$ such that $f(x \circ y) = (f(x) + f(y) - 1) \vee 0$. The automorphism f is unique.*

A nilpotent t-norm $\circ$ is strictly increasing whenever possible—that is, for $x, y, z \in (0, 1)$ with $y < z$, then $x \circ y < x \circ z$ if $x \circ z \neq 0$.

Among the examples of t-norms listed earlier, the nilpotent t-norms are the Łukaciewisz t-norm, the generalized Łukaciewisz t-norms for $a > 0$, and the Yager t-norms.

In the case of strict (convex Archimedean) t-norms we have shown that $Aut(\mathbb{I}, \circ) = f^{-1}R^{+}f \subseteq Aut(\mathbb{I})$ where f is a generator of $\circ$. We can now show that these are the only convex t-norms with such automorphism groups.

Proposition 21 *Let $\circ$ be any convex t-norm. Then $Aut(\mathbb{I}, \circ) = f^{-1}\mathbb{R}^{+}f$ for some $f \in Aut(\mathbb{I})$ if and only if $\circ$ is a strict t-norm.*

Proof. Suppose that $Aut(\mathbb{I}, \circ) = f^{-1}\mathbb{R}^{+}f$ and suppose that for some $a \in (0, 1)$, $a \circ a = a$. Then for any $b \in (0, 1)$, there is an element $g \in Aut(\mathbb{I}, \circ)$ such that $g(a) = b$, namely $g = f^{-1}rf$ where $r = \ln f(b) / \ln f(a)$. Thus

$$b \circ b = g(a) \circ g(a) = g(a \circ a) = g(a) = b,$$

so that $\circ$ is idempotent. But the only idempotent t-norm is min, and $Aut(\mathbb{I}, \min) = Aut(\mathbb{I}) \neq f^{-1}\mathbb{R}^{+}f$. Thus $a \circ a < a$ for all $a \in (0, 1)$, and $\circ$ is Archimedean. The t-norm is not nilpotent, by Corollary 18, and thus it is strict. ■

We cannot conclude that the function f in the proposition is a generator of $\circ$. We show in Section 3.2 that there are automorphisms in $N(\mathbb{R}^{+})$ besides those in $\mathbb{R}^{+}$. It follows that there are convex t-norms $\circ$ different from multiplication for which $Aut(\mathbb{I}, \circ) = \mathbb{R}^{+}$. In that case, $f = \text{id}$ but id is clearly not a generator for $\circ$.

2.3 Negations

In this section, we will prove results analogous to those for t-norms in the previous sections. An element $f \in Map(\mathbb{I})$ has **order** n if $f^{n} = 1$, and n is the smallest such positive integer. If no such integer exists, the element has **infinite** order. All the elements of $Aut(\mathbb{I})$ have infinite order except 1 which has order 1. All antiautomorphisms are either of order two or of infinite order. Antiautomorphisms of order 2 are called **involutions**.

We will only concern ourselves with what are usually called strong negations. We will call them simply negations.

Definition 22 *A* ***negation*** *(or involution) on* $\mathbb{I}$ *is an antiautomorphism of* $\mathbb{I}$ *of order two.*

Negations will generally be denoted by small Greek letters. Thus a negation is an order-reversing, one-to-one mapping β of $\mathbb{I}$ onto $\mathbb{I}$ such that $\beta(\beta(x)) = x$, or equivalently such that $\beta^2 = 1$.

We reserve the notation α to denote the negation given by $\alpha(x) = 1 - x$. It is a trivial fact that conjugates of negations are negations, but it is a little less trivial and a bit surprising that every negation is a conjugate of α by an automorphism.

Theorem 23 *Let* β *be a negation in* $Map(\mathbb{I})$ *and define* f *by*

$$f(x) = \frac{\alpha\beta(x) + x}{2}$$

Then $f \in Aut(\mathbb{I})$, *and* $\beta = f^{-1}\alpha f$. *Moreover,* $g^{-1}\alpha g = \beta$ *if and only if* $gf^{-1} \in Z(\alpha)$.

This will be proved in greater generality in Section 5.1, along with a generalization of the fact that the centralizer $Z(\beta)$ of a negation $\beta = f^{-1}\alpha f$ is the group

$$\begin{aligned} Z(\beta) &= \left\{ \frac{\beta g\beta + g}{2} : g \in Aut(\mathbb{I}) \right\} \\ &= f^{-1}Z(\alpha)f = f^{-1} \left\{ \frac{\alpha g\alpha + g}{2} : g \in Aut(\mathbb{I}) \right\} f \end{aligned}$$

(see Theorem 2).

An automorphism f such that $\beta = f^{-1}\alpha f$ is a **generator** of β. So every negation has a generator, and we know when two elements of $Aut(\mathbb{I})$ give the same negation. This theorem seems to be due to Trillas [41] who takes as generators functions from $[0, 1]$ to $[0, \infty]$. Notice, however, that using elements of $Map(\mathbb{I})$ as generators of t-norms, t-conorms, and negations enables us to use the language of group theory and to involve only functions on $[0, 1]$. Also, we are automatically provided with natural operations between generators, being elements of the group $Map(\mathbb{I})$. To us, this is an important mathematical point.

Theorem 24 *Let* β *be a negation and let* f *be a generator of* β. *The map* $\beta \to Z(\alpha)f$ *is a one-to-one correspondence between the negations of* $\mathbb{I}$ *and the set of right cosets of the centralizer* $Z(\alpha)$ *of* α.

Consider two systems $(\mathbb{I}, \beta)$ and $(\mathbb{I}, \gamma)$ where β and γ are negations. They are **isomorphic** if there is a map $h \in Aut(\mathbb{I})$ with $h(\beta(x)) = \gamma h(x)$, that is if $h\beta = \gamma h$, or equivalently if $\beta = h^{-1}\gamma h$. Let f and g be generators of β and γ, respectively. If h is an isomorphism, then $hf^{-1}\alpha f = g^{-1}\alpha gh$ which means that

$$f^{-1}\alpha f = h^{-1}g^{-1}\alpha gh = (gh)^{-1}\alpha gh$$

Therefore, f and gh generate the same negation, and so $zf = gh$ for some $z \in Z(\alpha)$. Thus $h \in g^{-1}Z(\alpha)f$. It is easy to check that elements of $g^{-1}Z(\alpha)f$ are isomorphisms $(\mathbb{I}, \beta) \to (\mathbb{I}, \gamma)$. We have the following theorem.

Theorem 25 *Let β and γ be negations with generators f and g, respectively. Then the set of isomorphisms from $(\mathbb{I}, \beta)$ to $(\mathbb{I}, \gamma)$ is $g^{-1}Z(\alpha)f$. In particular, $g^{-1}f$ is an isomorphism from $(\mathbb{I}, \beta)$ to $(\mathbb{I}, \gamma)$.*

Note that $Z(\alpha)$ plays a role for negations analogous to that of $\mathbb{R}^+$ for strict t-norms. See Corollary 13. If we call two negations β and γ **isomorphic** if $(\mathbb{I}, \beta) \approx (\mathbb{I}, \gamma)$, then the previous theorem says in particular that any two negations are isomorphic. We have the following special cases.

Corollary 26 *The set of isomorphisms from $(\mathbb{I}, \beta)$ to $(\mathbb{I}, \alpha)$ is the right coset $Z(\alpha)f$ of $Z(\alpha)$. In particular, the generator f of β is an isomorphism from $(\mathbb{I}, \beta)$ to $(\mathbb{I}, \alpha)$.*

For any negation β, $Aut(\mathbb{I}, \beta) = Z(\beta)$. Noting that $Z(\beta) = f^{-1}Z(\alpha)f$, we have

Corollary 27 *If f is a generator of β, then $Aut(\mathbb{I}, \beta) = f^{-1}Z(\alpha)f$.*

Since $z \to f^{-1}zf$ is an isomorphism from $Z(\alpha) = Aut(\mathbb{I}, \alpha)$ to $f^{-1}Z(\alpha)f = Aut(\mathbb{I}, \beta)$, we get

Corollary 28 *For any two negations β and γ, $Aut(\mathbb{I}, \beta) \approx Aut(\mathbb{I}, \gamma)$.*

Of course this last corollary follows also because the two systems $(\mathbb{I}, \beta)$ and $(\mathbb{I}, \gamma)$ are isomorphic. The upshot of all this is that furnishing $\mathbb{I}$ with any negation yields a system isomorphic to that gotten by furnishing $\mathbb{I}$ with the negation $\alpha : x \to 1 - x$.

2.4 DeMorgan systems

Let $\circ$ be a t-norm and β a negation. Then $\diamond$ defined by $x \diamond y = \beta(\beta(x) \circ \beta(y))$ defines a binary operation on $[0, 1]$ called a **t-conorm**. It has the following characterizing properties.

- $0 \diamond x = x$.
- $x \diamond y = y \diamond x$.
- $(x \diamond y) \diamond z = x \diamond (y \diamond z)$.
- $\diamond$ is increasing in each variable.

If $\diamond$ is convex and for $x \in (0,1)$, $x \diamond x > x \diamond 0 = x$, the t-conorm $\diamond$ is **Archimedean**. If $\diamond$ satisfies these properties, then $\circ$ defined by $x \circ y = \beta(\beta(x) \diamond \beta(y))$, using any negation β, is a t-norm, and $x \diamond y = \beta(\beta(x) \circ \beta(y))$. If a t-norm and t-conorm are related in this way by the negation β, then $(\mathbb{I}, \circ, \beta, \diamond)$ is a **deMorgan system**, and the t-norm $\circ$ and the t-conorm $\diamond$ are said to be **dual to one another** via the negation β.

Now suppose that

$$q : (\mathbb{I}, \circ, \beta, \diamond) \to (\mathbb{I}, \triangle, \gamma, \bigtriangledown)$$

is an isomorphism. This means that $q \in Aut(\mathbb{I})$ and the following hold.

$$\begin{aligned} q(x \circ y) &= q(x) \triangle q(y) \\ q(\beta(x)) &= \gamma(q(x)) \\ q(x \diamond y) &= q(x) \bigtriangledown q(y) \end{aligned}$$

But since $x \diamond y = \beta(\beta(x) \circ \beta(y))$ and $x \bigtriangledown y = \gamma(\gamma(x) \triangle \gamma(y))$, if the first two equations hold, then so does the third. Therefore to be an isomorphism, q need only be required to satisfy the first two conditions. That is, isomorphisms from $(\mathbb{I}, \circ, \beta, \diamond)$ to $(\mathbb{I}, \triangle, \gamma, \bigtriangledown)$ are the same as isomorphisms from $(\mathbb{I}, \circ, \beta)$ to $(\mathbb{I}, \triangle, \gamma)$. We will also call these systems **deMorgan systems**.

3 DeMorgan systems with strict t-norms

In this section, *we are going to restrict ourselves to strict t-norms.* Their duals are called **strict t-conorms**. DeMorgan systems with nilpotent t-norms will be investigated in Section 4.

Suppose that f is a generator of the strict t-norm $\circ$, β is a negation, and

$$\begin{aligned} x \diamond y &= \beta(\beta(x) \circ \beta(y)) \\ &= \beta f^{-1}(f\beta(x))(f\beta(y)) \\ &= (f\beta)^{-1}(f\beta(x))(f\beta(y)) \end{aligned}$$

Thus the anti-automorphism $f\beta$ and multiplication determine the strict t-conorm $\diamond$. In general, an anti-automorphism g of $\mathbb{I}$ is a **cogenerator** of a strict t-conorm $\diamond$ if $x \diamond y = g^{-1}(g(x)(g(y))$. It should be clear that every t-conorm has a cogenerator, and it is easy to check that g and h are cogenerators of the same t-conorm if and only if $g = rh$ for some $r \in \mathbb{R}^+$.

For notational reasons, we are going to adorn our operators with their generators. Thus a strict t-norm $\circ$ will be written $\circ_f$, meaning that f is a generator of $\circ$. Ordinary multiplication is $\circ_r$ for any $r \in \mathbb{R}^+$, but that will be denoted as usual by $\cdot$. A conorm with cogenerator h will be denoted $\diamond_h$. Finally, α_f denotes the negation with generator f. We write α_1 simply as α. So a deMorgan system looks like $(\mathbb{I}, \circ_f, \alpha_g, \diamond_{f\alpha_g})$, or simply $(\mathbb{I}, \circ_f, \alpha_g)$.

To determine the isomorphisms q from $(\mathbb{I}, \circ_f, \alpha_g)$ to $(\mathbb{I}, \circ_u, \alpha_v)$, we just note that such a q must be an isomorphism from $(\mathbb{I}, \circ_f)$ to $(\mathbb{I}, \circ_u)$ and from $(\mathbb{I}, \alpha_g)$ to $(\mathbb{I}, \alpha_v)$. Therefore, from Theorems 14 and 25, we get the following theorem.

Theorem 29 *The set of isomorphisms from* $(\mathbb{I}, \circ_f, \alpha_g)$ *to* $(\mathbb{I}, \circ_u, \alpha_v)$ *is the set*

$$\left(u^{-1}\mathbb{R}^+ f\right) \cap \left(v^{-1}Z(\alpha)g\right)$$

This intersection may be empty, of course. That is the case when the equation $u^{-1}rf = v^{-1}zg$ has no solution for $r > 0$ and $z \in Z(\alpha)$. A particular example of this is the case where $f = g = u = 1$, $v \notin Z(\alpha)$, and $v\left(\frac{1}{2}\right) = \frac{1}{2}$. Then $r = v^{-1}z$ with $r > 0$ and $z \in Z(\alpha)$. But then

$$r\left(\frac{1}{2}\right) = v^{-1}z\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^r = \frac{1}{2}$$

Thus $r = 1$, and so $v = z$. But $v \notin Z(\alpha)$. So there are deMorgan systems $(\mathbb{I}, \circ_f, \alpha_g)$ and $(\mathbb{I}, \circ_u, \alpha_v)$ which are not isomorphic. When two deMorgan systems are isomorphic, the isomorphism is unique and the situation is this.

Theorem 30 $(\mathbb{I}, \circ_f, \alpha_g) \approx (\mathbb{I}, \circ_u, \alpha_v)$ *if and only if* $(\mathbb{I}, \circ_u, \alpha_v) = (\mathbb{I}, \circ_{fh}, \alpha_{gh})$ *for some* $h \in Aut(\mathbb{I})$*, in which case* h^{-1} *is the only such isomorphism. In particular,* $(\mathbb{I}, \circ_f, \alpha_g) \approx (\mathbb{I}, \cdot, \alpha_{gf^{-1}})$*.*

Proof. It is easy to check that h^{-1} is an isomorphism from $(\mathbb{I}, \circ_u, \alpha_v)$ to $(\mathbb{I}, \circ_{fh}, \alpha_{gh})$. If k is such an isomorphism, then $k = u^{-1}rf = v^{-1}zg$ for some $r \in \mathbb{R}^+$ and $z \in Z(\alpha)$. Thus $u = rfk^{-1}$ and $v = zgk^{-1}$ and so $(\mathbb{I}, \circ_u, \alpha_v) = (\mathbb{I}, \circ_{fk^{-1}}, \alpha_{gk^{-1}})$. If k were distinct from h^{-1}, then kh would be a non-trivial automorphism of $(\mathbb{I}, \circ_f, \alpha_g)$. But by Proposition 3, and Theorem 25, this is impossible. ■

One implication of this theorem, taking $f = g$, is that the theory of the deMorgan system $(\mathbb{I}, \circ_f, \alpha_f)$ is the same as that of $(\mathbb{I}, \cdot, \alpha)$. More generally this holds for $(\mathbb{I}, \circ_f, \alpha_g)$ and $(\mathbb{I}, \cdot, \alpha_{gf^{-1}})$. This suggests that in applications of deMorgan systems, one may as well take the strict t-norm to be ordinary multiplication.

Corollary 31 $Aut((\mathbb{I}, \circ_f, \alpha_g)) = \{\mathrm{id}\}$*.*

Corollary 32 $(\mathbb{I}, \cdot, \beta) \approx (\mathbb{I}, \cdot, \gamma)$ *if and only if* $\gamma = r^{-1}\beta r$ *for some* $r \in \mathbb{R}^+$*.*

Among the negations $\gamma = r^{-1}\beta r$ for $r \in \mathbb{R}^+$ there is exactly one with a given fixed point $u \in (0, 1)$, so for example, there is a one-to-one correspondence between isomorphism classes of deMorgan systems with strict t-norms and de Morgan systems of the form $(\mathbb{I}, \cdot, \beta)$ where $\cdot$ is multiplication and β has fixed point $\frac{1}{2}$.

Taking $\beta = \alpha$ in this last Corollary, we see that $(\mathbb{I}, \cdot, \alpha) \approx (\mathbb{I}, \cdot, \gamma)$ if and only if $\gamma = r^{-1}\alpha r$ for some $r \in \mathbb{R}^+$. So deMorgan systems isomorphic to $(\mathbb{I}, \cdot, \alpha)$ are exactly those of the form $(\mathbb{I}, \cdot, \alpha_r)$ with $r \in \mathbb{R}^+$. Negations of the form $r^{-1}\alpha r$ are **Yager negations** [42]. Thus we can state

Corollary 33 *DeMorgan systems* $(\mathbb{I}, \cdot, \beta)$ *which are isomorphic to* $(\mathbb{I}, \cdot, \alpha)$ *are precisely those with* β *a Yager negation.*

We close this section with the following remark. The system $([0,1], \wedge, \vee, ')$, where $\wedge$, $\vee$, and $'$ are max, min, and $x' = 1 - x$ forms a deMorgan algebra in the usual lattice theoretic sense. If we replace $'$, which we have been denoting by α, by any other involution β, then the systems $([0,1], \wedge, \vee, ')$ and $([0,1], \wedge, \vee, \beta)$ are isomorphic. Isomorphisms between these algebras are exactly the isomorphisms between $(\mathbb{I}, \alpha)$ and $(\mathbb{I}, \beta)$. There are many and these are spelled out in Theorem 25. This suggests that in applications of deMorgan systems where $\wedge$ and $\vee$ are taken for the t-norm and t-conorm, respectively, the negation may as well be $\alpha(x) = 1 - x$.

3.1 Non-uniqueness of negations in strict deMorgan systems

We have noted that a deMorgan system $(\mathbb{I}, \circ, \beta, \diamond)$ is determined by the system $(\mathbb{I}, \circ, \beta)$. Of course, it is also determined by the system $(\mathbb{I}, \beta, \diamond)$. Is it determined by $(\mathbb{I}, \circ, \diamond)$? How unique is the negation in a deMorgan system? When is $(\mathbb{I}, \circ, \diamond)$ a reduct of a deMorgan system? The following lemma is straightforward and applies to both strict and nilpotent deMorgan systems.

Lemma 34 *If* $(\mathbb{I}, \circ, \beta, \diamond)$ *and* $(\mathbb{I}, \circ, \gamma, \diamond)$ *are deMorgan systems having the same t-norm and t-conorm, then* $\gamma\beta \in Aut\,(\mathbb{I}, \circ)$.

If $(\mathbb{I}, \circ, \beta, \diamond)$ and $(\mathbb{I}, \circ, \gamma, \diamond)$ are strict deMorgan systems, this means that $\gamma\beta \in f^{-1}\mathbb{R}^{+}f$ where f is a generator of $\circ$ (Corollary 15). From this it can be shown that $\eta = f\beta f^{-1}$ is a negation satisfying $r\eta r = \eta$ (where $r = f\gamma\beta f^{-1}$). Moreover, $\beta = f^{-1}\eta f$ and $\gamma = f^{-1}\eta r^{-1} f$. On the other hand, if η is a negation such that for some $r > 0$, $r\eta r = \eta$, then it is routine to check that for any t-norm $\circ_f$, $f^{-1}\eta f$, $f^{-1}\eta r f$ and $f^{-1}\eta r^{-1} f$ are negations that give the same t-conorm. The following example shows that there are nontrivial cases where this occurs, so in particular, a deMorgan system is not determined by the t-norm and t-conorm alone. In the example the corresponding deMorgan systems are isomorphic even though the negations are not equal, but this may not always be the case.

For a positive real number a, let $\eta_a(x) = e^{\frac{a}{\ln x}}$. Then η_a is a negation satisfying $r\eta_a r = \eta_a$ for all $r > 0$. And for any $f \in Aut\,(\mathbb{I})$, $f^{-1}\eta_a f$ and $f^{-1}\eta_a r f$ are negations which give the same t-conorm dual to the t-norm generated by f. It is easy to see that $\eta_a r = \eta_{\frac{a}{r}}$, so these two negations are of the same form—that is, $f^{-1}\eta_a r f = f^{-1}\eta_{\frac{a}{r}} f$. We get the following: for any strict t-norm $\circ$ with generator f, and positive real numbers a and b, the t-conorms dual to $\circ$ via the negations $f^{-1}\eta_a f$ and $f^{-1}\eta_b f$ are the same.

Consider the deMorgan system $(\mathbb{I}, \cdot, \eta_a)$,where the t-norm is multiplication, and suppose it is isomorphic to $(\mathbb{I}, \cdot, \beta)$. The isomorphism must be some $r \in \mathbb{R}^{+}$ since it is an automorphism of $(\mathbb{I},\cdot)$ and $\beta=\eta_{ar^2}$. Moreover, for any $a,b\in\mathbb{R}^+$, $\sqrt{\frac{b}{a}}:(\mathbb{I},\cdot,\eta_a)\to(\mathbb{I},\cdot,\eta_b)$ is an isomorphism. We sum up.

Corollary 35 *The deMorgan systems $(\mathbb{I},\cdot,\eta_a)$ are all isomorphic. If $(\mathbb{I},\cdot,\eta_a)\approx(\mathbb{I},\cdot,\beta)$, then $\beta=\eta_r$ for some $r\in\mathbb{R}^+$.*

The negations η_a satisfy $r\eta_a r=\eta_a$ for *all* $r\in\mathbb{R}^+$. There are no other negations with this property. However, Professor Richard Bagby at New Mexico State University has constructed a large family of negations β such that $r\beta r=\beta$ for a *fixed* $r\neq 1$. Thus for each one of these negations β, the negations $\gamma=f^{-1}\beta f$ and $\delta=f^{-1}\beta rf$ produce the same t-conorm from the t-norm generated by f. It seems likely that for some such negations the deMorgan systems $(\mathbb{I},\circ_f,f^{-1}\beta f)$ and $(\mathbb{I},\circ_f,f^{-1}\beta rf)$ are not isomorphic even though the t-conorms are the same. They will be, however, whenever $\sqrt{r}\beta\sqrt{r}=\beta$ also holds. Constructing and somehow classifying all such β seems not to have been done.

Let $\circ$ and $\diamond$ be a strict t-norm and strict t-conorm with generator f and cogenerator g, respectively. When does there exist a negation β such that $\circ$ and $\diamond$ are dual with respect to β? This means finding a negation β such that

$$\beta(f^{-1}(f(\beta(x))f(\beta(y)))=g^{-1}(g(x)g(y))$$

This in turns means that $f\beta=rg$ for some $r>0$. The existence of such a β is equivalent to the existence of a negation in the set $f^{-1}\mathbb{R}^+g$. There may be many or there may be no such negations.

3.2 The normalizer of $\mathbb{R}^+$ and Richman t-norms

Were $\mathbb{R}^+$ a normal subgroup in $Aut\,(\mathbb{I})$, this would put a natural group structure on the set of all (strict) t-norms. This is not the case, however. That is, the normalizer $\{f\in Aut\,(\mathbb{I}):f^{-1}\mathbb{R}^+f\ =\mathbb{R}^+\}$ of $\mathbb{R}^+$ in the group $Aut\,(\mathbb{I})$ is not all of $Aut\,(\mathbb{I})$. But the set of t-norms with generators in the normalizer of $\mathbb{R}^+$ in the group $Aut\,(\mathbb{I})$ does carry a group structure, and our primary concerns are identifying this group and giving explicit formulas for the corresponding set of t-norms.

We will proceed a little more generally, viewing $\mathbb{R}^+$ as a subgroup of $Map\,(\mathbb{I})$. Anti-automorphisms f are generators of t-conorms, given by $x\diamond y=f^{-1}\left(f\left(x\right)f\left(y\right)\right)$. Let $N\left(\mathbb{R}^+\right)$ be the **normalizer** of $\mathbb{R}^+$ in the group $Map\,(\mathbb{I})$; that is,

$$N\left(\mathbb{R}^+\right)=\left\{f\in Map\,(\mathbb{I}):f^{-1}\mathbb{R}^+f\ =\mathbb{R}^+\right\}$$

The **centralizer** $Z\left(\mathbb{R}^+\right)$ of $\mathbb{R}^+$ in $Map\,(\mathbb{I})$ is the set

$$Z\left(\mathbb{R}^+\right)=\left\{f\in Map\,(\mathbb{I}):fr=rf\text{ for all }r\in\mathbb{R}^+\right\}$$

The following lemma is crucial. We give its proof. It is short and imparts the flavor of the proofs we will omit.

Lemma 36 $Z(\mathbb{R}^+) = \mathbb{R}^+$.

Proof. If $f \in M$ and $fr = rf$ for all $r \in \mathbb{R}^+$, then for $x \in (0,1)$, $fr(x) = f(x^r) = (f(x))^r$. For any $y \in (0,1)$, $y = x^k$ for some positive real number k. So $f(xy) = f(xx^k) = f(x^{k+1}) = (f(x))^{k+1} = f(x) f(x)^k = f(x) f(y)$. Thus f preserves multiplication on $(0,1)$. If f is an anti-automorphism, then there is an $x \in (0,1)$ with $f(x) \neq x$, and f has a fixed point y. Let $x^r = y$. Then $f(x^r) = x^r = (f(x))^r \neq x^r$. So $f \in A$, and $f(xy) = f(x) f(y)$. By Corollary 16, $f(x) = x^r$ for some r. This means that $f \in \mathbb{R}^+$. ■

The proof of the following theorem is purely group theoretic. See [38] for details.

Theorem 37 $N(\mathbb{R}^+) = \left\{ f \in Map(\mathbb{I}) : f(x) = e^{-c(-\ln x)^k},\ c > 0,\ k \neq 0 \right\}$.

The group structures of $N(\mathbb{R}^+)$ and $N(\mathbb{R}^+)/\mathbb{R}^+$ are described in the following corollary.

Corollary 38 *The normalizer $N(\mathbb{R}^+)$ of $\mathbb{R}^+$ in $Map(\mathbb{I})$ is isomorphic to the group*

$$\{(c,k) : c > 0, k \neq 0\}$$

with multiplication given by

$$(c', k')(c, k) = \left(c' c^{k'}, k'k\right).$$

The subgroup $\mathbb{R}^+$ corresponds to $\{(c,1) : c \in \mathbb{R}^+\}$ and $N(\mathbb{R}^+)/\mathbb{R}^+$ to $\{(1,k) : k \neq 0\}$. Thus the natural group structure carried by the set of norms and conorms with generators in $N(\mathbb{R}^+)$ is the multiplicative group $\mathbb{R}^*$ of the nonzero real numbers.

The group $N(\mathbb{R}^+)$ splits: $N(\mathbb{R}^+) = Z(\mathbb{R}^+) \times K$, with $Z(\mathbb{R}^+)$ normal, corresponding to $\mathbb{R}^+$, and K isomorphic to the multiplicative group $\mathbb{R}^*$ of the nonzero real numbers.

To find the norms and conorms with generators in $N(\mathbb{R}^+)$ for $f \in N(\mathbb{R}^+)$, we must compute $f^{-1}(f(x) f(y))$. If

$$f(x) = e^{-c(-\ln x)^k},$$

then

$$f^{-1}(x) = e^{-\left(-\frac{\ln x}{c}\right)^{\frac{1}{k}}}$$

and

$$\begin{aligned} f^{-1}(f(x) f(y)) &= f^{-1}\left(e^{-c(-\ln x)^k - c(-\ln y)^k}\right) \\ &= e^{-\left(-\frac{-c(-\ln x)^k - c(-\ln y)^k}{c}\right)^{\frac{1}{k}}} \\ &= e^{-\left((-\ln x)^k + (-\ln y)^k\right)^{\frac{1}{k}}} \end{aligned}$$

Of course, the quantity c does not appear in the formula since the t-norm or t-conorm generated by f is independent of c. It is straightforward to check the items in the following corollary.

Corollary 39 *The t-norms with generators in $N(\mathbb{R}^+)$ are given by*

$$x \circ y = e^{-\left((-\ln x)^k + (-\ln y)^k\right)^{\frac{1}{k}}}$$

with k positive. The t-conorms with generators in $N(\mathbb{R}^+)$ are given by

$$x \diamond y = e^{-\left((-\ln x)^k + (-\ln y)^k\right)^{\frac{1}{k}}}$$

with k negative. Ordinary multiplication is the identity element of the group of t-norms and t-conorms. That is, for $k = 1$,

$$x \circ y = e^{-(-\ln x - \ln y)} = xy.$$

The t-norms correspond to the positive elements $\mathbb{R}^+$ of the group $\mathbb{R}^*$, given by its parameter k in $e^{-\left((-\ln x)^k + (-\ln y)^k\right)^{\frac{1}{k}}}$. Thus with each such t-norm with parameter k, there is associated the t-conorm with parameter $-k$. It is an easy calculation to verify the following corollary.

Corollary 40 *The negations in N are precisely the elements $e^{-c(-\ln x)^{-1}} = e^{\frac{c}{\ln x}}$, that is, the elements in N with parameter $k = -1$. For $k > 0$, the t-norm*

$$e^{-\left((-\ln x)^k + (-\ln y)^k\right)^{\frac{1}{k}}}$$

is dual to the t-conorm

$$e^{-\left((-\ln x)^{-k} + (-\ln y)^{-k}\right)^{\frac{1}{-k}}}$$

with respect to any of the negations $\eta(x) = e^{\frac{c}{lnx}}$.

All the generators of the t-norms and t-conorms $e^{-\left((-\ln x)^k + (-\ln y)^k\right)^{\frac{1}{k}}}$ are in $N(\mathbb{R}^+)$. This is because these t-norms and t-conorms do have generators in $N(\mathbb{R}^+)$, namely $e^{-c(-\ln x)^k}$, with the positive k giving norms and the negative k giving t-conorms. Generators are unique up to composition with an element of $\mathbb{R}^+$, and since the group $N(\mathbb{R}^+)$ contains $\mathbb{R}^+$, our claim follows. If a t-norm with generator f is dual to a t-conorm via a negation η, then $f\eta$ is a generator of the t-conorm. (See [18], page 745, for example.) Thus if a t-norm and t-conorm with generators in $N(\mathbb{R}^+)$ are dual, then they must be dual with respect to a negation in $N(\mathbb{R}^+)$. But for a generator $f(x) = e^{-c(-\ln x)^k}$ of a t-norm, and negation $e^{d/\ln x}$, $f\eta(x) = e^{(-c/d)(-\ln x)^{-k}}$. The following sums it up.

Corollary 41 *Let k and r be positive. Then the t-norm $e^{-\left((-\ln x)^k + (-\ln y)^k\right)^{\frac{1}{k}}}$ is dual to the t-conorm $e^{-\left((-\ln x)^{-r} r + (-\ln y)^{-r}\right)^{\frac{1}{-r}}}$ if and only if $r = k$, in which case they are dual with respect to precisely the negations $e^{\frac{c}{\ln x}}$ in $N(\mathbb{R}^+)$.*

4 DeMorgan systems with nilpotent t-norms

In this section, *we are going to restrict ourselves to nilpotent t-norms.* Their duals are called **nilpotent t-conorms**. We will take for generators the isomorphism of the nilpotent t-norm with the Łukaciewisz t-norm. Suppose that f is a generator of the nilpotent t-norm $\circ$, β is a negation, and

$$\begin{aligned} x \diamond y &= \beta(\beta(x) \circ \beta(y)) \\ &= \beta f^{-1}\left(((f\beta(x)) + (f\beta(y)) - 1) \vee 0\right) \\ &= (f\beta)^{-1}\left(((f\beta(x)) + (f\beta(y)) - 1) \vee 0\right) \end{aligned}$$

Thus the antiautomorphism $f\beta$ and the Łukaciewisz t-norm determine the nilpotent t-conorm $\diamond$. In general, an antiautomorphism g of $\mathbb{I}$ is a **cogenerator** of a nilpotent t-conorm $\diamond$ if $x \diamond y = g^{-1}\left(((g(x)) + (g(y)) - 1) \vee 0\right)$.

It should be clear that every t-conorm has a cogenerator, and the cogenerator is unique.

As with strict t-norms, we will adorn our operators with their generators. Thus a nilpotent t-norm $\circ$ will be written $\blacktriangle_f$, meaning that f is a generator of $\circ$. A conorm with generator h will be denoted $\blacktriangledown_h$. Finally, α_f denotes the negation with generator f. We write $\blacktriangle_{\text{id}}$ and α_{id} simply as $\blacktriangle$ and α. So a nilpotent deMorgan system looks like $(\mathbb{I}, \blacktriangle_f, \alpha_g, \blacktriangledown_{f\alpha_g})$ or simply $(\mathbb{I}, \blacktriangle_f, \alpha_g)$.

To determine the isomorphisms q from $(\mathbb{I}, \blacktriangle_f, \alpha_g)$ to $(\mathbb{I}, \blacktriangle_u, \alpha_v)$, we just note that such a q must be an isomorphism from $(\mathbb{I}, \blacktriangle_f)$ to $(\mathbb{I}, \blacktriangle_u)$ and from $(\mathbb{I}, \alpha_g)$ to $(\mathbb{I}, \alpha_v)$. Therefore, from Theorems 14 and 25, we get the following theorem.

Theorem 42 *The set of isomorphisms from* $(\mathbb{I}, \blacktriangle_f, \alpha_g)$ *to* $(\mathbb{I}, \blacktriangle_u, \alpha_v)$ *is the set*

$$\left\{u^{-1}f\right\} \cap \left(v^{-1}Z(\alpha)g\right)$$

This intersection may be empty, of course. That is the case when the equation $u^{-1}f = v^{-1}zg$ has no solution for $z \in Z(\alpha)$. A particular example of this is the case where $f = g = u = \text{id}$ and $v \notin Z(\alpha)$. Then $\text{id} = v^{-1}z$ with $z \in Z(\alpha)$. So $v = z$. But $v \notin Z(\alpha)$. So there are nilpotent deMorgan systems $(\mathbb{I}, \blacktriangle_f, \alpha_g)$ and $(\mathbb{I}, \blacktriangle_u, \alpha_v)$ which are not isomorphic. When two deMorgan systems are isomorphic, the isomorphism is unique and the situation is this.

Theorem 43 $(\mathbb{I}, \blacktriangle_f, \alpha_g) \approx (\mathbb{I}, \blacktriangle_u, \alpha_v)$ *if and only if* $(\mathbb{I}, \blacktriangle_u, \alpha_v) = (\mathbb{I}, \blacktriangle_{fh}, \alpha_{gh})$ *for some* $h \in Aut(\mathbb{I})$*, in which case* h^{-1} *is the only such isomorphism. In particular,* $(\mathbb{I}, \blacktriangle_f, \alpha_g) \approx (\mathbb{I}, \blacktriangle, \alpha_{gf^{-1}})$.

Proof. It is easy to check that h^{-1} is an isomorphism from $(\mathbb{I}, \blacktriangle_f, \alpha_g)$ to $(\mathbb{I}, \blacktriangle_{fh}, \alpha_{gh})$. If k is such an isomorphism, then $k = u^{-1}f = v^{-1}zg$ for some $z \in Z(\alpha)$. Thus $u = fk^{-1}$ and $v = zgk^{-1}$ and so $(\mathbb{I}, \blacktriangle_u, \alpha_v) = (\mathbb{I}, \blacktriangle_{fk^{-1}}, \alpha_{gk^{-1}})$. If k were distinct from

h^{-1}, then kh would be a non-trivial automorphism of $(\mathbb{I}, \blacktriangle_f, \alpha_g)$, and in particular, of $(\mathbb{I}, \blacktriangle_f)$. But by Corollary 18 this is impossible. ■

One implication of this theorem, taking $f = g$, is that the theory of the deMorgan system $(\mathbb{I}, \blacktriangle_f, \alpha_f)$ is the same as that of $(\mathbb{I}, \blacktriangle, \alpha)$. More generally this holds for $(\mathbb{I}, \blacktriangle_f, \alpha_g)$ and $(\mathbb{I}, \blacktriangle, \alpha_{gf^{-1}})$. This suggests that in applications of deMorgan systems, one may as well take the nilpotent t-norm to be the Łukaciewisz t-norm.

Corollary 44 $Aut((\mathbb{I}, \blacktriangle_f, \alpha_g)) = \{\mathrm{id}\}$.

Corollary 45 $(\mathbb{I}, \blacktriangle, \beta) \approx (\mathbb{I}, \blacktriangle, \gamma)$ *if and only if* $\gamma = \beta$.

Thus there is a one-to-one correspondence between isomorphism classes of deMorgan systems with nilpotent t-norms and deMorgan systems of the form $(\mathbb{I}, \blacktriangle, \beta)$ where $\blacktriangle$ is the Łukaciewisz t-norm. In contrast to the strict t-norm case, the negation connecting a given t-norm and t-conorm is unique. This follows directly from Lemma 34 since $\gamma\beta \in Aut\,(\mathbb{I}, \blacktriangle) = \{\mathrm{id}\}$.

4.1 Negations and nilpotent t-norms

A nilpotent t-norm distributes over infinite meets and joins. These facts are needed in the proof of the following theorem which reveals an important direct connection between nilpotent t-norms and negations. See [21] for details of the proof.

Theorem 46 *A nilpotent t-norm* $\triangle$ *on* $\mathbb{I}$ *determines a negation* $\eta_\triangle$ *by the equation*

$$\eta_\triangle (x) = \bigvee \{y \in [0, 1] : x \triangle y = 0\}$$

Definition 47 *If* $\triangle$ *is a binary operation on a lattice* $\mathbb{L}$ *with* 0*, an element* x^* *in* $\mathbb{L}$ *is the* $\triangle$-***pseudocomplement*** *of an element* x *if* $x \triangle y = 0$ *exactly when* $y \leq x^*$.

Theorem 46 says that for any nilpotent t-norm $\triangle$ on $\mathbb{I}$, the function $\eta_\triangle$ that gives the $\triangle$-pseudocomplement $\eta_\triangle (x)$ is a (strong) negation. It is easy to see that a t-norm $\triangle$ must be nilpotent in order for $\eta_\triangle$ to be a negation. As we shall soon see, every negation is the $\triangle$-pseudocomplement of some nilpotent t-norm $\triangle$. This will give two ways to represent negations—as $\alpha_\gamma = \gamma^{-1}\alpha\gamma$ for automorphisms γ of $\mathbb{I}$ (or more generally, as $\eta_\gamma = \gamma^{-1}\eta\gamma$ for fixed η), and as the $\triangle$-pseudocomplement $\eta_\triangle$ of a nilpotent t-norm $\triangle$.

In Section 2.2.2 we showed that any two nilpotent t-norms $\circ$ and $\triangle$ determine isomorphic systems $(\mathbb{I}, \circ)$ and $(\mathbb{I}, \triangle)$—that is, there is an automorphism f of $\mathbb{I}$ such that $f(x \circ y) = f(x) \triangle f(y)$ for all $x, y \in \mathbb{I}$. Moreover, given any nilpotent t-norm $\triangle$, and any automorphism f of $\mathbb{I}$, the binary operation $\triangle_f$ defined by $x \triangle_f y = f^{-1}(f(x) \triangle f(y))$ is again a nilpotent t-norm.

The connection between the representations $\eta_f = f^{-1}\eta f$ $(f \in \mathrm{Aut}\,(\mathbb{I}))$ and $\eta_\triangle$ $(\triangle \in \mathrm{Nilp}\,(\mathbb{I}))$ for negations is the following.

Theorem 48 *Choose any nilpotent t-norm* $\triangle$ *and let* $\eta_{\triangle}$ *be the* $\triangle$*-pseudocomplement. Then for each* $f \in \operatorname{Aut}(\mathbb{I})$

$$\left(\eta_{\triangle}\right)_f = \eta_{\triangle_f}$$

4.2 The group of nilpotent t-norms on $\mathbb{I}$

The map $f \mapsto \blacktriangle_f$ where $x \blacktriangle_f y = f^{-1}(f(x) \blacktriangle f(y))$ gives a one-to-one correspondence between the set of automorphisms of $\mathbb{I}$ and the set of nilpotent t-norms. The inverse correspondence is $\triangle \mapsto f_{\blacktriangle}^{\triangle}$ where $f_{\blacktriangle}^{\triangle}(x) \mapsto \left(\eta_{\blacktriangle}\eta_{\triangle}(x) + x\right)/2$ with $\eta_{\blacktriangle}$ and $\eta_{\triangle}$ the $\blacktriangle$-pseudocomplement and $\triangle$-pseudocomplement, respectively. This one-to-one correspondence puts a group structure on the set of nilpotent t-norms, namely,

$$\blacktriangle_f * \blacktriangle_g = x \blacktriangle_{fg} y$$

Example 49 *The subgroup corresponding to the group* $\mathbb{R}^+$ *of nonnegative reals is the one-parameter family of nilpotent t-norms*

$$x \blacktriangle_r y = ((x^r + y^r - 1) \vee 0)^{\frac{1}{r}}$$

for $r > 0$*, with* $\blacktriangle_r * \blacktriangle_s = \blacktriangle_{rs}$*.*

The subgroup corresponding to the normalizer of $\mathbb{R}^+$ *in* $\operatorname{Aut}(\mathbb{I})$ *(see Section 3.2) is the two-parameter family of nilpotent t-norms*

$$x \triangle_{c,k} y = e^{-\left(\frac{1}{c}\left(-\ln\left(\left(e^{-c(-\ln x)^k} + e^{-c(-\ln y)^k} - 1\right) \vee 0\right)\right)\right)^{\frac{1}{k}}}$$

for $c, k > 0$ *with* $\triangle_{c,k} * \triangle_{b,h} = \triangle_{bc^h,kh}$*.*

4.3 Stone and Boolean systems

The lattice $\mathbb{I}$ with the additional operations provided by a t-norm, t-conorm, and negation or other unary operations can satisfy properties reminiscent of axioms for deMorgan, Stone and Boolean algebras, and we name certain systems accordingly.

Definition 50 *Let* $(\mathbb{I}, \triangle, \triangledown, *)$ *be a system with t-norm* $\triangle$*, t-conorm* $\triangledown$*, and a decreasing unary operation* $*$*. We say that* $(\mathbb{I}, \triangle, \triangledown, *)$ *is a* ***Stone system*** *if* $*$ *is a* $\triangle$*-pseudocomplement—that is,*

$$x \triangle y = 0 \text{ if and only if } y \leq x^*$$

and if $*$ *also satisfies the identity*

$$x^* \triangledown x^{**} = 1$$

for all elements x *in* $[0,1]$*. (In this case,* $*$ *is a* $(\triangle, \triangledown)$***-complement*** *on its image—that is,* $x \triangle x^* = 0$ *and* $x \triangledown x^* = 1$ *for* x *in the image of* $*$*.) We say that* $(\mathbb{I}, \triangle, \triangledown, *)$ *is a* ***weak Boolean*** *system if* $*$ *is a* $\triangle$*-pseudocomplement, and* $(x \triangle y)^* = x^* \triangledown y^*$ *and* $(x \triangledown y)^* = x^* \triangle y^*$ *for all* $x, y \in \mathbb{I}$*. We call* $(\mathbb{I}, \triangle, \triangledown, *)$ *a* ***Boolean system*** *if it is both a Stone system and a deMorgan system.*

The preceding definition applies to strict t-norms and t-conorms as well as to nilpotent ones, but the situation in the strict case is relatively trivial. If $\triangle$ is a strict t-norm, then the $\triangle$-pseudocomplement is given by $0^{*_\triangle} = 1$ and $x^{*_\triangle} = 0$ for $x \neq 0$, and for any t-conorm $\bigtriangledown$, $(\mathbb{I}, \triangle, \bigtriangledown, *_\triangle)$ is a Stone system. Note that the image of $*_\triangle$ is the two-element Boolean algebra, and on this image, $\triangle = \wedge$, $\bigtriangledown = \vee$ and $*_\triangle$ is the complement. Since this $\triangle$-pseudocomplement is not a negation, there are no Boolean systems with a strict t-norm.

It is well known that a convex, Archimedean t-norm $\triangle$ on $\mathbb{I}$ is nilpotent exactly when there is a pair $x, y \in (0,1)$ with $x \triangle y = 0$, and a convex, Archimedean t-conorm $\bigtriangledown$ on $\mathbb{I}$ is nilpotent exactly when there is a pair $x, y \in (0,1)$ with $x \bigtriangledown y = 1$. This leads to the following necessary condition when $*$ is continuous.

Theorem 51 *If $(\mathbb{I}, \triangle, \bigtriangledown, *)$ is a Stone system in which $*$ is continuous, then both $\triangle$ and $\bigtriangledown$ are nilpotent.*

Theorem 46 established that for a nilpotent t-norm $\triangle$, the $\triangle$-pseudocomplement

$$\eta_\triangle(x) = \bigvee \{y \in [0,1] : x \triangle y = 0\}$$

is a negation. If η is any negation on $\mathbb{I}$ or $\mathbb{I}^{[2]}$ and $x \triangle \eta(x) = 0$, then the dual to $\triangle$ via the negation η satisfies

$$x \bigtriangledown \eta(x) = \eta(\eta(x) \triangle \eta(\eta(x))) = \eta(\eta(x) \triangle x) = \eta(0) = 1.$$

This yields the following theorems.

Theorem 52 *If $\triangle$ is a nilpotent t-norm, then $(\mathbb{I}, \triangle, \bigtriangledown, *)$ is a Boolean system if and only if $* = \eta_\triangle$ and $x \bigtriangledown y = \eta_\triangle(\eta_\triangle(x) \triangle \eta_\triangle(y))$.*

Theorem 53 *If $\triangle$ is a nilpotent t-norm, then $(\mathbb{I}, \triangle, \bigtriangledown, *)$ is a Stone system if and only if $* = \eta_\triangle$ and for all x,*

$$\eta_\triangle(x) \geq \eta_\bigtriangledown(x) = \bigwedge \{y \in [0,1] : x \bigtriangledown y = 1\}$$

If $(\mathbb{I}, \triangle, \bigtriangledown, *)$ is a Boolean system then equality holds (but this is not a sufficient condition since $\eta_\bigtriangledown$ does not determine $\bigtriangledown$).

An **MV-algebra** is an algebra $(A, \oplus, \neg, 0)$ with a binary operation, a unary operation, and a constant 0 for which $(A, \oplus, 0)$ is an abelian monoid, $\neg$ is an involution, and the equation

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$$

holds for all $x, y \in A$. These axioms are satisfied for $A = [0,1]$, $\oplus$ a nilpotent t-conorm, and $\neg$ the negation naturally determined by $\oplus$. The completeness theorem of Łukaciewisz, Wajsberg, Chang and others (see Chang [10]) states that the variety of MV-algebras is generated by the algebra $([0,1], \blacktriangledown, \alpha, 0)$ where $\blacktriangledown$ is the Łukaciewisz t-conorm and $\alpha(x) = 1 - x$. The usual lattice structure on $[0,1]$ follows from the axioms for an MV-algebra, so that $([0,1], \blacktriangledown, \alpha, 0)$ determines the Boolean system $(\mathbb{I}, \blacktriangle, \alpha)$ and conversely.

In Section 2.2.2, we stated two basic facts about nilpotent convex Archimedean t-norms: any two are isomorphic—that is, any two algebras $(\mathbb{I}, \triangle)$ are isomorphic, and each has a trivial automorphism group $\operatorname{Aut}(\mathbb{I}, \triangle)$. These two facts carry over immediately to Boolean systems, since the $\triangle$-pseudocomplement and the dual t-conorm are both naturally determined by $\triangle$.

Example 54 *The Boolean systems corresponding to the positive reals in the group of nilpotent t-norms with base point $\blacktriangle$ are of the form $\left(\mathbb{I}, \blacktriangle_r, \eta_{\blacktriangle_r}, \blacktriangledown_r\right)$ with*

$$\begin{aligned} x \blacktriangle_r y &= ((x^r + y^r - 1) \vee 0)^{\frac{1}{r}} \\ \eta_{\blacktriangle_r} &= (1 - x^r)^{\frac{1}{r}} \\ x \blacktriangledown_r y &= ((x^r + y^r))^{\frac{1}{r}} \wedge 1 \end{aligned}$$

The one-parameter family $\blacktriangle_r$ of t-norms is well-known, and these t-norms are often paired with their duals relative to α. Members of the one-parameter family $\blacktriangledown_r$ of t-conorms are known as Yager t-conorms, the Yager t-norms being their duals relative to α.

Given a nilpotent t-norm there is exactly one Boolean system with this t-norm. Given a negation η, the number of nilpotent t-norms $\triangle$ such that $\eta = \eta_\triangle$ is the same as the number of automorphisms in the centralizer of $\eta_\triangle$, namely, the power of the continuum. See [21].

Corollary 55 *For each negation η there are uncountably many Boolean systems $(\mathbb{I}, \triangle, \eta)$.*

5 Averaging operators on the unit interval

In Section 2.3 we used the map $\eta \mapsto \frac{\alpha\eta + \mathrm{id}}{2}$ from the set of negations to the group of automorphisms of $\mathbb{I}$, and in Section 2.1 we used the map $f \mapsto \frac{\alpha f \alpha + f}{2}$ from the group of automorphisms of $\mathbb{I}$ to the centralizer of α. In both cases we used an operation, averaging, that is not inherent in the lattice structure of $\mathbb{I}$. An average also provides a (continuous) scaling of the unit interval that is not provided by the lattice structure. Following our philosophy of working entirely within a well-defined algebraic system, we now consider averaging operators and add such an operator to a deMorgan system. We use the following definition which is a variant of those in the references [27, 33, 2, 16, 4, 11, 5, 43, 30].

Definition 56 *An* ***averaging operator*** *on* $\mathbb{I}$ *is a binary operation* $\dotplus : \mathbb{I}^2 \to \mathbb{I}$ *satisfying for all* $x, y \in [0,1]$,

1. $x \dotplus y = y \dotplus x$ ($\dotplus$ *is* commutative).
2. $y < z$ *implies* $x \dotplus y < x \dotplus z$ ($\dotplus$ *is* strictly increasing *in each variable*).
3. $x \dotplus y \leq c \leq x \dotplus z$ *implies there exists* $w \in [y, z]$ *with* $x \dotplus w = c$ ($\dotplus$ *is* convex, *i.e. continuous*).
4. $x \dotplus x = x$ ($\dotplus$ *is* idempotent).
5. $(x \dotplus y) \dotplus (z \dotplus w) = (x \dotplus z) \dotplus (y \dotplus w)$ ($\dotplus$ *is* bisymmetric).

A system $(\mathbb{I}, \dotplus)$ will be called a **mean system**. The following properties of an averaging operator are well-known.

Proposition 57 *Let* $\dotplus$ *be an averaging operator. Then for each* $x, y \in [0,1]$,

1. $x \wedge y \leq x \dotplus y \leq x \vee y$*—that is, the average of* x *and* y *lies* between x *and* y.
2. *The function* $A_x : \mathbb{I} \to [x \dotplus 0, x \dotplus 1] : y \longmapsto x \dotplus y$ *is an isomorphism—that is,* A_x *is an increasing function that is both one-to-one, and onto.*

The standard averaging operator is the arithmetic mean:

$$\mathrm{av}\,(x, y) = \frac{x + y}{2}.$$

Other examples include the power means and logarithmic means:

$$\begin{aligned} x \dotplus y &= \left(\tfrac{x^a + y^a}{2}\right)^{\frac{1}{a}} \\ x \dotplus y &= \log_a (a^x + a^y) \end{aligned}$$

Indeed, for any automorphism or antiautomorphism γ of $\mathbb{I}$,

$$x \dotplus y = \gamma^{-1}\left(\frac{\gamma(x) + \gamma(y)}{2}\right) = \gamma^{-1}\left(\mathrm{av}\left(\gamma(x), \gamma(y)\right)\right)$$

is an averaging operator.

This last example is universal—that is, the following representation theorem holds (see, for example, Aczel [4] page 287).

Theorem 58 *If* $\dotplus$ *is an averaging operator on* $\mathbb{I}$*, then there is a unique automorphism* f *of* $\mathbb{I}$ *such that*

$$f\left(x \dotplus y\right) = \frac{f(x) + f(y)}{2}$$

for all $x, y \in [0,1]$.

Thus every averaging operator on $[0,1]$ is isomorphic to the usual averaging operator on $[0,1]$—that is, the mean systems $\left(\mathbb{I},\dot{+}\right)$ and $(\mathbb{I},\mathrm{av})$ are isomorphic as algebras.

Corollary 59 *For any averaging operator* $\dot{+}$*, the automorphism group of* $(\mathbb{I},\dot{+})$ *has only one element.*

When an averaging operator is given by the formula

$$x\dot{+}y=f^{-1}\left(\frac{f(x)+f(y)}{2}\right)$$

for an automorphism f of $\mathbb{I}$, we will call f a **generator** of the operator $\dot{+}$ and write $\dot{+}=\dot{+}_f$. From the theorem above, the generator of an averaging operator is unique.

It is easy to see that if f,g are automorphisms [antiautomorphisms] of $\mathbb{I}$, and $\dot{+}$ is an averaging operator on $\mathbb{I}$, then $f\dot{+}g$ defined by $(f\dot{+}g)(x)=f(x)\dot{+}g(x)$ is again an automorphism [antiautomorphism] of $\mathbb{I}$.

5.1 Negations and averaging operators

Each averaging operator naturally determines a negation, with respect to which the averaging operator is self-dual. The proof of the following theorem is typical.

Theorem 60 *For each averaging operator* $\dot{+}$ *on* $\mathbb{I}$*, the equation*

$$x\dot{+}\eta(x)=0\dot{+}1$$

defines a negation $\eta=\eta_{\dot{+}}$ *on* $\mathbb{I}$ *with fixed point* $0\dot{+}1$.

Proof. Since $x\dot{+}0=0\dot{+}x\leq 0\dot{+}1\leq x\dot{+}1$, by Proposition 57, for each $x\in[0,1]$ there is a number $y\in[0,1]$ such that $x\dot{+}y=0\dot{+}1$, and since A_x is strictly increasing, there is only one such y for each x. Thus the equation defines a function $y=\eta(x)$. Clearly $\eta(0)=1$ and $\eta(1)=0$. Suppose $0\leq x<y\leq 1$. We know $x\dot{+}\eta(x)=y\dot{+}\eta(y)=0\dot{+}1$. If $\eta(x)\leq\eta(y)$, then $x\dot{+}\eta(x)<y\dot{+}\eta(x)\leq y\dot{+}\eta(y)$ which is not the case. Thus $\eta(x)>\eta(y)$ and η is a strictly decreasing function. Now $\eta(\eta(x))$ is defined by $\eta(x)\dot{+}\eta(\eta(x))=0\dot{+}1$. But also, $\eta(x)\dot{+}x=x\dot{+}\eta(x)=0\dot{+}1$. Thus, applying Proposition 57 to $\eta(x)$, we see that $\eta(\eta(x))=x$. It follows that η is a negation. If x is the fixed point of η, then $x=x\dot{+}x=x\dot{+}\eta(x)=0\dot{+}1$. ■

Theorem 61 *Every homomorphism between mean systems preserves the negations associated with these means.*

Proof. Suppose $f:(\mathbb{I},\dot{+}_1)\to(\mathbb{I},\dot{+}_2)$ is a homomorphism. Then

$$\begin{aligned} f(x)\dot{+}_2 f\left(\eta_{\dot{+}_1}(x)\right) &= f\left(x\dot{+}_1\eta_{\dot{+}_1}(x)\right)=f(0\dot{+}_1 1)\\ &= f(0)\dot{+}_2 f(1)=0\dot{+}_2 1 \end{aligned}$$

Thus $f\left(\eta_{\dot{+}_1}(x)\right) = \eta_{\dot{+}_2}(f(x))$. ■

For this reason, **mean systems with natural negation** $\left(\mathbb{I}, \dot{+}, \eta_{\dot{+}}\right)$ will be often be referred to simply as **mean systems**.

Corollary 62 *If f is the generator of $\dot{+}$, then $\eta_{\dot{+}} = f^{-1}\alpha f$.*

Proof. The generator f is an isomorphism between $(I, \dot{+})$ and $(\mathbb{I}, \mathrm{av})$, so from Theorem 61, we see that $f\eta_{\dot{+}} = \alpha f$. ■

Example 63 *For $x \dot{+} y = \dfrac{x+y}{2}$, $\eta_{\dot{+}}(x) = 1 - x$*

For $x \dot{+} y = \left(\dfrac{x^a + y^a}{2}\right)^{\frac{1}{a}}$, $\eta_{\dot{+}}(x) = (1 - x^a)^{\frac{1}{a}}$

For $x \dot{+}_a y = \log_a(x^a + y^a)$, $\eta_{\dot{+}_a}(x) = \log_a(1 + a - a^x)$ and $\lim_{a\to 1} \eta_{\dot{+}_a}(x) = 1 - x$

The following theorem shows that $\dot{+}$ is self-dual with respect to its natural negation —that is,

$$x \dot{+} y = \eta_{\dot{+}}\left(\eta_{\dot{+}}(y) \dot{+} \eta_{\dot{+}}(x)\right)$$

Theorem 64 *Let $\dot{+}$ be an averaging operator on $\mathbb{I}$. Then $\eta_{\dot{+}}$ is an antiautomorphism of the system $(\mathbb{I}, \dot{+})$. Moreover, it is the only antiautomorphism of $(\mathbb{I}, \dot{+})$.*

The following three theorems generalize Theorems 2, 22, and 23 of [18], where these theorems are proved for the arithmetic mean and the corresponding negation $\alpha(x) = 1 - x$. In the proofs of the following theorems, the bisymmetry plays a crucial role. See [21] for details.

Theorem 65 *Let $\dot{+}$ be an averaging operator on $\mathbb{I}$ and let η be the negation determined by the equation $x \dot{+} \eta(x) = 0 \dot{+} 1$. Then the centralizer $Z(\eta)$ of η is the set of elements of the form $\eta f \eta \dot{+} f$ for automorphisms f of $\mathbb{I}$. Moreover, if $f \in Z(\eta)$, then $\eta f \eta \dot{+} f = f$.*

Theorem 66 *If β is a negation, then $\beta = f^{-1}\eta f$ for f the automorphism of $\mathbb{I}$ defined by $f(x) = \eta\beta(x) \dot{+} x$, where $\dot{+}$ is any averaging operator such that $\eta = \eta_{\dot{+}}$. Moreover, $\beta = g^{-1}\eta g$ if and only if $gf^{-1} \in Z(\eta)$.*

Theorem 67 *Let $\dot{+}$ be an averaging operator on $\mathbb{I}$, and let η be the negation determined by the equation $x \dot{+} \eta(x) = 0 \dot{+} 1$. The map $\beta \longmapsto Z(\eta)(\eta\beta \dot{+} \mathrm{id})$ is a one-to-one correspondence between the negations of $\mathbb{I}$ and the set of right cosets of the centralizer $Z(\eta)$ of η.*

5.2 Averaging operators and nilpotent t-norms

It was observed in Theorem 60 that the negation generated by f is the same as the negation associated with the averaging operator $\dot{+}_f$—that is, $\alpha_f = \eta_{\dot{+}_f}$. A similar relationship holds for the nilpotent t-norm $\blacktriangle_f$.

Proposition 68 *For an automorphism f of $\mathbb{I}$, the negations α_f, $\eta_{\blacktriangle_f}$, and $\eta_{\dot{+}_f}$ coincide, that is,*

$$x \blacktriangle_f y = 0 \quad \text{if and only if} \quad y \leq \alpha_f (x)$$

and

$$x \dot{+}_f \eta_{\blacktriangle_f} (x) = x \dot{+}_f \alpha_f (x) = 0 \dot{+}_f 1$$

There are a number of different averaging operators that give the same negation, namely one for each automorphism in the centralizer of that negation. The same can be said for nilpotent t-norms. However, there is a closer connection between averaging operators and nilpotent t-norms than a common negation. Given an averaging operator one can determine the particular nilpotent t-norm that has the same generator without knowing that generator, and conversely, as shown in the following theorem. This correspondence is a natural one—that is, it does not depend on the generator. Recall that for a nilpotent t-norm, the function defined by $\eta_\triangle (x) = \bigvee \{ y : x \triangle y = 0\}$ is a negation, Theorem 46.

Theorem 69 *The condition*

$$x \triangle y \leq z \quad \text{if and only if} \quad x \dot{+} y \leq z \dot{+} 1$$

determines a one-to-one correspondence between nilpotent t-norms and averaging operators, namely, given an averaging operator $\dot{+}$, define $\triangle_{\dot{+}}$ by

$$x \triangle_{\dot{+}} y = \bigwedge \{ z : x \dot{+} y \leq z \dot{+} 1\}$$

This correspondence preserves generators.

To describe the inverse correspondence, given a nilpotent t-norm $\triangle$, define a binary operation $*_\triangle$ by

$$x *_\triangle y = \bigvee \{ z : z \triangle z \leq x \triangle y\}$$

and define $\dot{+}_\triangle$ by

$$x \dot{+}_\triangle y = (x *_\triangle y) \wedge \left(\eta_\triangle \left(\eta_\triangle (x) *_\triangle \eta_\triangle (y)\right)\right)$$

The situation with strict t-norms is somewhat more complicated. We explore that in the next section.

5.3 DeMorgan systems with averaging operators

Given a system $\left(\mathbb{I}, \circ, \dot{+}\right)$, where $\circ$ is any t-norm and $\dot{+}$ is any averaging operator, we may as well view the system as a deMorgan system with an averaging operator $\left(\mathbb{I}, \circ, \eta_{\dot{+}}, \diamond, \dot{+}\right)$. We are interested in such systems with the compatibility property

$$(x \circ y) \dot{+} (x \diamond y) = x \dot{+} y$$

The family of t-norms $\triangle$ that satisfy the equation

$$(x \triangle y) + (x \bigtriangledown y) = x + y$$

for $x \bigtriangledown y = \alpha\left(\alpha\left(x\right) \triangle \alpha\left(x\right)\right)$ are called **Frank t-norms** [15]. Frank showed that this is the one-parameter family of t-norms of the form

$$x \triangle_{F_a} y = \log_a \left[1 + \frac{\left(a^x - 1\right)\left(a^y - 1\right)}{a - 1}\right], \; a > 0, \; a \neq 1$$

with limiting values

$$\begin{aligned} &x \triangle_{F_0} y = x \wedge y \\ &x \triangle_{F_1} y = xy \\ &x \triangle_{F_\infty} y = (x + y - 1) \vee 0 \end{aligned}$$

If $\circ$ is an arbitrary strict t-norm, and γ is an automorphism of $\mathbb{I}$, we will use the notation $\circ_\gamma$ for the t-norm defined by

$$x \circ_\gamma y = \gamma^{-1}\left(\gamma\left(x\right) \circ \gamma\left(y\right)\right)$$

Note that all the Frank t-norms for $0 < a < \infty$ are strict. The strict Frank t-norms are generated by functions of the form

$$\begin{aligned} &F_a\left(x\right) = \tfrac{a^x - 1}{a - 1}, \; a > 0, \; a \neq 1 \\ &F_1\left(x\right) = x \end{aligned}$$

A t-norm $\triangle$ is called **nearly Frank** [34] if there is an isomorphism $h : (\mathbb{I}, \triangle, \alpha) \to (\mathbb{I}, \triangle_F, \alpha)$ of deMorgan systems for some Frank t-norm $\triangle_F$—that is, for all $x \in [0, 1]$,

$$\begin{aligned} h\left(x \triangle y\right) &= h\left(x\right) \triangle_F h\left(y\right) \\ h\alpha\left(x\right) &= \alpha h\left(x\right) \end{aligned}$$

We generalize the notion of Frank t-norm to a deMorgan system with an arbitrary averaging operator $\dot{+}$:

Definition 70 *A system $\left(\mathbb{I}, \triangle, \eta, \bigtriangledown, \dot{+}\right)$ is a* ***Frank system*** *if $\triangle$ is a t-norm (nilpotent or strict), η is a negation, $\bigtriangledown$ is a t-conorm, $\dot{+}$ is an averaging operator, and the identities*

(1) $x \bigtriangledown y = \eta\left(\eta\left(x\right) \triangle \eta\left(y\right)\right)$ [$(\mathbb{I}, \triangle, \eta, \bigtriangledown)$ *is a deMorgan system.*]
(2) $x \dot{+} \eta\left(x\right) = 0 \dot{+} 1$ [$\left(\mathbb{I}, \eta, \dot{+}\right)$ *is a mean system with $\eta = \eta_{\dot{+}}$.*]
(3) $(x \triangle y) \dot{+} (x \bigtriangledown y) = x \dot{+} y$ [*The* ***Frank equation*** *is satisfied.*]

hold for all $x, y \in [0,1]$. A Frank system will be called a ***standard Frank system*** *if* $\dot{+} = \mathrm{av} = \dot{+}_{\mathrm{id}}$.

Note that in a standard Frank system $\left(\mathbb{I}, \triangle, \eta, \triangledown, \dot{+}\right)$, $\triangle$ is a Frank t-norm (nilpotent or strict) and $\eta = \alpha$. Also note that if $\dot{+}$ is generated by $h \in \mathrm{Aut}\,(\mathbb{I})$, the Frank equation is equivalent to

$$h\,(x \triangle y) + h\,(x \triangledown y) = h\,(x) + h\,(y)$$

If $\left(\mathbb{I}, \triangle, \eta, \triangledown, \dot{+}\right)$ is a Frank system, we will say the reduct $\left(\mathbb{I}, \triangle, \dot{+}\right)$ **determines a Frank system**, since η is determined algebraically by $\dot{+}$, and $\triangledown$ by η and $\triangle$.

Theorem 71 *The system $\left(\mathbb{I}, \circ, \eta, \triangledown, \dot{+}\right)$ is a Frank system if and only if it is isomorphic to a standard Frank system.*

Corollary 72 *For a nilpotent t-norm $\circ$ and an averaging operator $\dot{+}$, $\left(\mathbb{I}, \circ, \dot{+}\right)$ determines a Frank system if and only if $x \circ y = \bigwedge \left\{ z : x \dot{+} y \leq z \dot{+} 1 \right\}$.*

Corollary 73 *Let $R_s \subset \mathrm{Av}\,(\mathbb{I}) \times \mathrm{Strict}\,(\mathbb{I})$ be the relation defined by $\left(\dot{+}, \circ\right) \in R_s$ if and only if $\left(\mathbb{I}, \circ, \dot{+}\right)$ determines a strict Frank system. The following are equivalent.*

1. $\left(\dot{+}, \circ\right) \in R_s$
2. $\circ = \triangle_{F_a g}$ *and* $\dot{+} = \dot{+}_g$ *for some automorphism g of $\mathbb{I}$ and $a \in (0, \infty)$*
3. $\circ = \triangle_f$ *and* $\dot{+} = \dot{+}_{F_a^{-1} r f}$ *for some automorphism f of $\mathbb{I}$, $a, r \in (0, \infty)$.*

Thus $\left(\mathbb{I}, \triangle_f, \dot{+}_g\right)$ determines a strict Frank system if and only if f and g are related by $g \in F_a^{-1} \mathbb{R}^+ f$ for some a. Note that for every strict Archimedean, convex t-norm $\circ = \triangle_f$ there is a two-parameter family of Frank systems

$$F_a^{-1} r f : \left(\mathbb{I}, \triangle_f, \dot{+}_{F_a^{-1} r f}\right) \approx \left(\mathbb{I}, \triangle_{F_a}, \mathrm{av}\right)$$

and for every averaging operator $\dot{+} = \dot{+}_g$ there is a one-parameter family of strict Frank systems

$$g : \left(\mathbb{I}, \triangle_{F_a g}, \dot{+}_g\right) \approx \left(\mathbb{I}, \triangle_{F_a}, \mathrm{av}\right)$$

Also, for every nilpotent Archimedean, convex t-norm $\circ = \blacktriangle_\gamma$ there is a unique Frank system

$$\gamma : \left(\mathbb{I}, \blacktriangle_\gamma, \dot{+}_\gamma\right) \approx \left(\mathbb{I}, \blacktriangle, \mathrm{av}\right)$$

Thus every system of the form $(\mathbb{I}, \triangle)$ or $\left(\mathbb{I}, \dot{+}\right)$ is a reduct of one or more Frank systems. However, not every deMorgan system can be extended to a Frank system. The following theorems identify those that can. A nilpotent deMorgan system is called a **Boolean system** (see Section 4.3) if the negation is the one naturally determined by the t-norm.

Theorem 74 *A deMorgan system with nilpotent t-norm can be extended to a Frank system if and only if the system is Boolean. A deMorgan system* $(\mathbb{I}, \circ, \eta)$ *with strict t-norm* $\circ$ *can be extended to a Frank system if and only if there exists* $a \in \mathbb{R}^+$ *such that for* $f, g \in \mathrm{Aut}(\mathbb{I})$ *with* $\circ = \triangle_f$ *and* $\eta = \alpha_g$

$$F_a^{-1}\mathbb{R}^+ f \cap Z(\alpha) g \neq \emptyset$$

In this case,

$$F_a^{-1}\mathbb{R}^+ f \cap Z(\alpha) g = \{h\}$$

and the Frank system is

$$\left(\mathbb{I}, \triangle_f, \alpha_g, \dot{+}_h\right) = \left(\mathbb{I}, \triangle_{F_a h}, \alpha_h, \dot{+}_h\right)$$

Moreover, there is at most one such a*. The t-norm in the Frank system is nearly Frank if and only if* g *is in the centralizer of* α*.*

The intersection $F_a^{-1}\mathbb{R}^+ f \cap Z(\alpha) g$ may be empty for all $a > 0$. An example is given in [21].

6 Interval-valued fuzzy sets

For ordinary fuzzy set theory, the basic structure on $[0,1]$ is its lattice structure, coming from its order $\leq$. Subsequent operations on $[0,1]$ are operations on the bounded lattice $\mathbb{I} = ([0,1], \leq, 0, 1)$. Now consider fuzzy sets with interval values. The interval $[0,1]$ is replaced by the set $\{(a,b) : a, b \in [0,1],\ a \leq b\}$. The element (a,b) is just the *pair* with $a \leq b$. In lattice theory, there is a standard notation for this set: $[0,1]^{[2]}$. So if S is the universal set, then our new fuzzy sets are mappings $A : S \to [0,1]^{[2]}$. Now comes the crucial question. With what structure should $[0,1]^{[2]}$ be endowed? Again, lattice theory provides an answer. Use componentwise operations coming from the operations on $[0,1]$. For example, $(a,b) \leq (c,d)$ if $a \leq c$ and $b \leq d$, which gives the usual lattice max and min operations

$$(a,b) \vee (c,d) = (a \vee c, b \vee d)$$
$$(a,b) \wedge (c,d) = (a \wedge c, b \wedge d)$$

The resulting structure again has the standard notation $\mathbb{I}^{[2]}$. That is, $\mathbb{I}^{[2]}$ is the set $[0,1]^{[2]}$ with componentwise operations. This is a fundamental lattice theoretical construction: from a lattice L, form $L^{[2]}$ and use componentwise operations. The resulting lattice has many of the same properties as the original lattice. In particular, if L is a complete distributive lattice, then so is $L^{[2]}$. The proposal here is to use $\mathbb{I}^{[2]}$ as the basic building block for interval valued fuzzy set theory.

One negation on $\mathbb{I}^{[2]}$ is given by $(a,b)' = (b', a')$ where $x' = 1 - x$. With this negation, $\mathbb{I}^{[2]}$ becomes a deMorgan algebra $(\mathbb{I}^{[2]}, ', 0, 1)$. This in turn yields a deMorgan algebra for the set of all interval valued fuzzy sets with the operations

$$\begin{aligned}(A \wedge B)(s) &= A(s) \wedge B(s) \\ (A \vee B)(s) &= A(s) \vee B(s) \\ A'(s) &= (A(s))'\end{aligned}$$

6.1 Automorphisms of $\mathbb{I}^{[2]}$

A good share of the theory of fuzzy sets is concerned with endowing $\mathbb{I}$ with additional structure such as t-norms, t-conorms, and negations other than the usual min, max, and the negation $x \to 1 - x$. A basic problem for $\mathbb{I}^{[2]}$ is deciding on the appropriate definitions. For example, what should a t-norm on $\mathbb{I}^{[2]}$ be? Before addressing that problem, we first examine the set of automorphisms and antiautomorphisms of $\mathbb{I}^{[2]}$. This is fundamental for representation theorems for certain t-norms, t-conorms, and negations. We have no choice about what automorphisms and antiautomorphisms are: they are automorphisms and antiautomorphisms of our basic structure $\mathbb{I}^{[2]}$.

Definition 75 *An **automorphism** of $\mathbb{I}^{[2]}$ is a one-to-one map f from $[0,1]^{[2]}$ onto itself such that $f(x) \leq f(y)$ if and only if $x \leq y$. An **antiautomorphism** is a one-to-one map f from $[0,1]^{[2]}$ onto itself such that $f(x) \leq f(y)$ if and only if $x \geq y$.*

The set of all automorphisms of $\mathbb{I}^{[2]}$ is denoted $Aut(\mathbb{I}^{[2]})$ and the set of all automorphisms and antiautomorphisms is denoted $Map(\mathbb{I}^{[2]})$. Both of these are groups under composition of maps, and $Aut(\mathbb{I}^{[2]})$ is a normal subgroup of index 2 in $Map(\mathbb{I}^{[2]})$. Except for the identity, all the elements of $Aut(\mathbb{I}^{[2]})$ are of infinite order. Antiautomorphisms are of order two or of infinite order. These are trivial but pertinent facts. Antiautomorphisms of order two are **involutions**, also called **negations**, and that set is denoted $Neg(\mathbb{I}^{[2]})$. The map α given by $\alpha(a,b) = (1-b, 1-a)$ is a negation, as is $f^{-1}\alpha f$ for any $f \in Map(\mathbb{I}^{[2]})$. It turns out that there are no others. If f is an automorphism of $\mathbb{I}$, that is, a one-to-one map of $[0,1]$ onto itself such that $f(x) \leq f(y)$ if and only if $x \leq y$, then $(a,b) \to (f(a), f(b))$ is an automorphism of $\mathbb{I}^{[2]}$. It turns out that there are no others. The following lemma is crucial in proving this result.

In the plane, $[0,1]^{[2]}$ is a triangle inside the unit square. Each leg of this triangle is mapped onto itself by automorphisms.

Lemma 76 *Let $A = \{(0,x) : x \in [0,1]\}$, $B = \{(x,1) : x \in [0,1]\}$, and $C = \{(x,x) : x \in [0,1]\}$. If $f \in Aut(\mathbb{I}^{[2]})$ then $f(A) = A$, $f(B) = B$, and $f(C) = C$.*

See [19] for details of the proof of this lemma.

Theorem 77 *Every automorphism f of $\mathbb{I}^{[2]}$ is of the form $f(a,b) = (g(a), g(b))$, where g is an automorphism of $\mathbb{I}$.*

Proof. Since f is an automorphism of C, it induces an automorphism g of $\mathbb{I}$, namely $(g(x), g(x)) = f(x, x)$. Now $f(0,1) = (0,1)$ since $f(A) = A$ and $f(B) = B$. Thus

$$\begin{aligned} f(a,b) &= f(a,a) \vee (f(b,b) \wedge f(0,1)) \\ &= (g(a), g(a)) \vee ((g(b), g(b)) \wedge (0,1) \\ &= (g(a), g(b)) \end{aligned}$$

■

So automorphisms of $\mathbb{I}^{[2]}$ are of the form $(a,b) \to (f(a), f(b))$ where f is an automorphism of $\mathbb{I}$. We will use f to denote both the automorphism of $\mathbb{I}$ and the corresponding automorphism of $\mathbb{I}^{[2]}$. The following theorem is clear.

Theorem 78 $Aut(\mathbb{I}) \approx Aut(\mathbb{I}^{[2]})$.

Let α be the antiautomorphism of $\mathbb{I}$ given by $\alpha(a) = 1 - a$. Then $(a,b) \to (\alpha(b), \alpha(a))$ is an antiautomorphism of $\mathbb{I}^{[2]}$ which we also denote by α. If g is an antiautomorphism of $\mathbb{I}^{[2]}$, then $g = \alpha f$ for the automorphism $f = \alpha g$. Now

$$\begin{aligned} g(a,b) &= \alpha f(a,b) \\ &= \alpha(f(a), f(b)) \\ &= (\alpha f(b), \alpha f(a)) \\ &= (g(b), g(a)) \end{aligned}$$

We have

Corollary 79 *Every antiautomorphism of* $\mathbb{I}^{[2]}$ *is of the form* $(a,b) \to (g(b), g(a))$, *where* g *is an antiautomorphism of* $\mathbb{I}$.

Theorem 80 $Map(\mathbb{I}) \approx Map(\mathbb{I}^{[2]})$.

Due to this isomorphism, many algebraic properties of the systems on $\mathbb{I}$ and $\mathbb{I}^{[2]}$ are similar. The logics associated with the two systems are intrinsically different, however (see [20, 31] and Section 7 of this paper). Also the deMorgan systems in these two settings differ algebraically, as we will see in Section 6.4.

A lot of facts about automorphisms and antiautomorphisms now follow from the facts in Section 2. In particular, we have

Theorem 81 *The involutions of* $\mathbb{I}^{[2]}$ *are precisely the involutions* $\{f^{-1}\alpha f : f \in Aut(\mathbb{I}^{[2]})\}$.

6.2 t-norms on $\mathbb{I}^{[2]}$

Our first problem here is that of *defining* t-norms. There is a natural embedding of $\mathbb{I}$ into $\mathbb{I}^{[2]}$, namely $c \to (c, c)$. This is how $\mathbb{I}^{[2]}$ generalizes $\mathbb{I}$. Instead of specifying a number c (identified with (c, c)) as a degree of belief, an expert specifies an interval (a, b) with $a \leq b$. So no matter how a t-norm is defined on $\mathbb{I}^{[2]}$, it should induce a t-norm on this copy of $\mathbb{I}$ in it.

A t-norm should be increasing in each variable, just as in the case for $\mathbb{I}$. On $\mathbb{I}$ this is equivalent to the conditions $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$ and $x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z)$. But just increasing in each variable will not yield these distributive laws on $\mathbb{I}^{[2]}$. However, these distributive laws do imply increasing in each variable.

Now for the boundary conditions. The two elements $(0, 0)$ and $(1, 1)$ are the bounds of the lattice $\mathbb{I}^{[2]}$. These two elements, along with $(0, 1)$, play a special role in the lattice, as they are the only elements of the lattice fixed by all automorphisms. Since t-norms on $\mathbb{I}^{[2]}$ are to generalize t-norms on $\mathbb{I}$, we certainly want $(1, 1) \circ (a, b) = (a, b)$ for all $(a, b) \in \mathbb{I}^{[2]}$. It will follow that $(0, 0) \circ (a, b) = (0, 0)$, but what about the element $(0, 1)$? How is it to behave? In analogy, it is natural to require that $(0, 1) \circ (a, b) = (0, b)$, in particular that $(0, 1) \circ (0, 1) = (0, 1)$. We are led to the following definition.

Definition 82 *A commutative, associative binary operation* $\circ$ *on* $\mathbb{I}^{[2]}$ *is a* ***t-norm*** *if for all* $x, y, z \in [0, 1]^{[2]}$ *and* $(a, b) \in [0, 1]^{[2]}$

1. $C \circ C \subseteq C$, *where* $C = \{(c, c) : c \in [0, 1]\}$
2. $x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$
3. $x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z)$
4. $(1, 1) \circ x = x$
5. $(0, 1) \circ (a, b) = (0, b)$.

Several additional useful properties follow immediately for a t-norm $\circ$ on $\mathbb{I}^{[2]}$.

1. $\circ$ is increasing in each variable.
2. $x \circ y \leq x \wedge y$
3. $(0, 0) \circ x = (0, 0)$
4. $(0, b) \circ x = (0, e)$ for some e.
5. The restriction of $\circ$ to $A = \{(0, a) : a \in [0, 1]\}$, $B = \{(b, 1) : b \in [0, 1]\}$, or $C = \{(c, c) : c \in [0, 1]\}$ induces a t-norm on $\mathbb{I}$.

To see that $\circ$ is increasing, suppose that $y \leq z$. Then

$$x \circ z = x \circ (y \vee z) = (x \circ y) \vee (x \circ z).$$

For the fourth property, note that $(0, b) \circ (c, d) \leq (0, 1) \circ (c, d) = (0, d)$, and observe that all elements less than or equal to $(0, d)$ are of the form $(0, e)$ for some e.

With this definition, t-norms on $\mathbb{I}^{[2]}$ are a natural extension of t-norms on $\mathbb{I}$.

Theorem 83 *Every t-norm $\diamond$ on $\mathbb{I}^{[2]}$ is of the form*

$$(a, b) \diamond (c, d) = (a \circ c, b \circ d)$$

where $\circ$ is a t-norm on $\mathbb{I}$.

Most of the t-norms we will consider are convex in the following sense. As is the case for $\mathbb{I}$, the convex t-norms on $\mathbb{I}^{[2]}$ are exactly the continuous t-norms.

Definition 84 *A binary operation $\circ$ on $\mathbb{I}$ or $\mathbb{I}^{[2]}$ is* ***convex*** *if given $x \circ y \leq c \leq u \circ v$, there exists an r between x and u and an x between y and v such that $c = r \circ s$.*

In the case of convex binary operations we can weaken condition 5 of the definition of t-norm.

Theorem 85 *A commutative, associative, convex binary operation on is a t-norm if for all $x, y, z \in [0, 1]^{[2]}$*

1. *$C \circ C \subseteq C$, where $C = \{(c, c) : c \in [0, 1]\}$*
2. *$x \circ (y \vee z) = (x \circ y) \vee (x \circ z)$*
3. *$x \circ (y \wedge z) = (x \circ y) \wedge (x \circ z)$*
4. *$(1, 1) \circ x = x$*
5. *$(0, 1) \circ (0, 1) = (0, 1)$.*

Proof. Suppose $\circ$ is a commutative, associative, convex binary operation on $\mathbb{I}^{[2]}$. Then for any $x \in \mathbb{I}^{[2]}$, $x \circ 0 = (x \wedge 1) \circ 0 = (x \circ 0) \wedge (1 \circ 0) = (x \circ 0) \wedge 0 = 0$. If $x \circ x = x$, then for any $y \in \mathbb{I}^{[2]}$, $x \circ 0 = 0 \leq x \wedge y \leq x = x \circ x$, so there is an element z with $0 \leq z \leq x$ and $x \circ z = x \wedge y$. Then

$$x \wedge y = x \circ z = (x \circ x) \circ z = x \circ (x \circ z) = x \circ (x \wedge y) \leq x \circ y.$$

But $x \circ y = x \circ (y \wedge 1) = (x \circ y) \wedge (x \circ 1) \leq x$ and $x \circ y = (x \wedge 1) \circ y = (x \circ y) \wedge (1 \circ y) \leq y$. Thus $x \circ y = x \wedge y$. Now, since $(0, 1) \circ (0, 1) = (0, 1)$, we have

$$(0, 1) \circ (a, b) = (0, 1) \wedge (a, b) = (0, b).$$

■

Now we define Archimedean, strict, and nilpotent t-norms on $\mathbb{I}^{[2]}$ just as for t-norms on $\mathbb{I}$. In the context of a t-norm $\circ$ we will write $x^n = \overbrace{x \circ x \circ \cdots \circ x}^{n \text{ times}}$.

Definition 86 *A t-norm* $\circ$ *on* $\mathbb{I}^{[2]}$ *is* ***Archimedean*** *if given any* $x, y \in \mathbb{I}^{[2]}$ *with* $x, y \notin \{(0,0),(0,1),(1,1)\}$*, there is a positive integer* n *with* $x^n \leq y$*. A convex Archimedean t-norm is* ***strict*** *if* $x \circ x = 0$ *only for* $x = 0$ *and* ***nilpotent*** *otherwise.*

The characterization of convex Archimedean t-norms on $\mathbb{I}^{[2]}$ is analogous to that for $\mathbb{I}$.

Proposition 87 *A convex t-norm* $\circ$ *on* $\mathbb{I}^{[2]}$ *is Archimedean if and only if it satisfies* $x \circ x < x$ *for all* $x \in \mathbb{I}^{[2]} \backslash \{(0,0),(0,1),(1,1)\}$*.*

A t-norm on $\mathbb{I}$ is nilpotent if and only if for each $x \in [0,1]$, there is a positive integer n such that $x^n \in \{0,1\}$. The corresponding condition for a nilpotent t-norm on $\mathbb{I}^{[2]}$ is that for each $x \in [0,1]^{[2]}$ there is a positive integer n such that $x^n \in \{(0,0),(0,1),(1,1)\}$.

It is easy to see that, in the notation of Theorem 83, a convex t-norm $\diamond$ is Archimedean, strict, or nilpotent if and only if the t-norm $\circ$ is Archimedean, strict, or nilpotent, respectively. In many respects, the theory of t-norms on $\mathbb{I}^{[2]}$ as we have defined them is reduced to the theory of t-norms on $\mathbb{I}$.

6.3 Negations and t-conorms on $\mathbb{I}^{[2]}$

An antiautomorphism f such that $f(f(x)) = x$ is an **involution**, or **negation**. The map α given by $\alpha(a,b) = (1-b, 1-a)$ is a negation, as is $f^{-1}\alpha f$ for any automorphism f, and there are no others. Antiautomorphisms interchange $(0,0)$ and $(1,1)$. Just as for $\mathbb{I}$, we define a t-conorm to be the dual of a t-norm with respect to some negation.

Definition 88 *Let* $\triangle$ *be a binary operation and* η *a negation on* $\mathbb{I}^{[2]}$*. The* ***dual of*** $\triangle$ ***with respect to*** η *is the binary operation* $\triangledown$ *given by*

$$a \triangledown b = \eta(\eta(a) \triangle \eta(b))$$

If $\triangle$ *is a t-norm, then* $\triangledown$ *is called a* ***t-conorm****.*

See [19] for proofs of the results in this section.

Theorem 89 *Every t-conorm* $\triangledown$ *on* $\mathbb{I}^{[2]}$ *is of the form*

$$(a,b) \triangledown (c,d) = (a \triangledown c, b \triangledown d)$$

where $\triangledown$ *is a t-conorm on* $\mathbb{I}$*.*

The proof follows by duality from the proof for t-norms. Thus the theory of t-conorms and negations on $\mathbb{I}^{[2]}$ has also been reduced to that theory on $\mathbb{I}$.

The following theorem gives properties characterizing t-conorms.

Theorem 90 *A commutative, associative binary operation* $\triangledown$ *on* $\mathbb{I}^{[2]}$ *is a t-conorm if and only if for all* $x, y, z, (a, b) \in \mathbb{I}^{[2]}$

1. $C \triangledown C = C$, *where* $C = \{(x, x) : x \in [0, 1]\}$
2. $x \triangledown (y \vee z) = (x \triangledown y) \vee (x \triangledown z)$
3. $x \triangledown (y \wedge z) = (x \triangledown y) \wedge (x \triangledown z)$
4. $(0, 0) \triangledown x = x$
5. $(0, 1) \triangledown (a, b) = (a, 1)$.

Definition 91 *A t-conorm* $\triangledown$ *is* ***convex*** *if given* $x \triangledown y \leq c \leq u \triangledown v$, *there exists an* r *between* x *and* u *and an* s *between* y *and* v *such that* $c = r \triangledown s$.

Theorem 92 *Every convex t-conorm* $\diamond$ *on* $\mathbb{I}^{[2]}$ *is of the form*

$$(a, b) \diamond (c, d) = (a \triangledown c, b \triangledown d)$$

where $\triangledown$ *is a t-conorm on* $\mathbb{I}$.

Note that for a convex binary operation $\triangledown$ on $\mathbb{I}$ or $\mathbb{I}^{[2]}$, if $x \triangledown x = x$, then $x \triangledown y = x \vee y$ for all y. It follows that for a convex binary operation, condition 5 of the characterization of t-conorms can be replaced by the weaker condition $(0, 1) \triangledown (0, 1) = (0, 1)$.

Implications are defined in terms of t-norms, t-conorms, and negations, so one can also develop the theory of implications for $\mathbb{I}^{[2]}$ from that of $\mathbb{I}$. In conclusion, the theory of deMorgan systems on $\mathbb{I}^{[2]}$ is a natural extension of the theory of deMorgan systems on $\mathbb{I}$.

6.4 Stone systems on $\mathbb{I}^{[2]}$

In Section 4.3 we considered Boolean, weak Boolean and Stone systems on $\mathbb{I}$. The extensions of these notions to systems on $\mathbb{I}^{[2]}$ reveals some of the algebraic differences between these systems.

Definition 93 *Let* $\mathbb{L}$ *be either the lattice* $\mathbb{I}$ *or* $\mathbb{I}^{[2]}$ *with t-norm* $\triangle$, *t-conorm* $\bigtriangledown$, *and a decreasing unary operation* $*$. *We say that* $(\mathbb{L}, \triangle, \bigtriangledown, *)$ *is a* ***Stone system*** *if* $*$ *is a* $\triangle$*-pseudocomplement—that is,*

$$x \triangle y = 0 \text{ if and only if } y \leq x^*$$

and if $*$ *also satisfies the identity*

$$x^* \bigtriangledown x^{**} = 1$$

*for all elements x in the lattice. (In this case, $*$ is a $(\triangle, \bigtriangledown)$-**complement** on its image—that is, $x \triangle x^* = 0$ and $x \bigtriangledown x^* = 1$ for x in the image of $*$.) We say that $(\mathbb{L}, \triangle, \bigtriangledown, *)$ is a **weak Boolean** system if $*$ is a $\triangle$-pseudocomplement, and $(x \triangle y)^* = x^* \bigtriangledown y^*$ and $(x \bigtriangledown y)^* = x^* \triangle y^*$ for all $x, y \in \mathbb{L}$. We call $(\mathbb{L}, \triangle, \bigtriangledown, *)$ a **Boolean system** if it is both a Stone system and a deMorgan system.*

See [21] for proofs of the following results.

Lemma 94 *Let $\triangle$ be a nilpotent t-norm on $\mathbb{I}^{[2]}$ and let $\circ$ be the t-norm on $\mathbb{I}$ such that $(a,b) \triangle (c,d) = (a \circ c, b \circ d)$. Then the $\triangle$-pseudocomplement of (a,b) in $\mathbb{I}^{[2]}$ is*

$$(a,b)^{*_\triangle} = (\eta_\circ(b), \eta_\circ(b))$$

where $\eta_\circ(b)$ is the $\circ$-pseudocomplement of b in $\mathbb{I}$.

Theorem 95 *A nilpotent system $(\mathbb{I}^{[2]}, \triangle, \bigtriangledown, *)$ is a Stone system if and only if $* = *_\triangle$ and the $\bigtriangledown$-pseudocomplement defined by $x^{*_\bigtriangledown} = \bigwedge \left\{ y \in \mathbb{I}^{[2]} : x \bigtriangledown y = 1 \right\}$ satisfies*

$$(b,b)^{*_\triangle} \geq (b,b)^{*_\bigtriangledown}$$

for all $(b,b) \in \mathbb{I}^{[2]}$.

Note that $*_\triangle$ is not a negation, but is the unique $\triangle$-pseudocomplement for $\mathbb{I}^{[2]}$, so in particular, a Stone system on $\mathbb{I}^{[2]}$ is never Boolean.

Theorem 96 *Suppose $\left(\mathbb{I}^{[2]}, \triangle, \bigtriangledown, *\right)$ is a Stone system, and let $(\mathbb{I}, \circ, \diamond, \eta_\circ)$ be the system on $\mathbb{I}$ satisfying*

$$\begin{aligned} (a,b) \triangle (c,d) &= (a \circ c, b \circ d) \\ (a,b) \bigtriangledown (c,d) &= (a \diamond c, b \diamond d) \\ (a,b)^* &= (\eta_\circ(b), \eta_\circ(b)) \end{aligned}$$

Then $(\mathbb{I}, \circ, \diamond, \eta_\circ)$ is a Stone system. Moreover, $\left(\mathbb{I}^{[2]}, \triangle, \bigtriangledown, *\right)$ is a weak Boolean system if and only if the corresponding system $(\mathbb{I}, \circ, \diamond, \eta_\circ)$ is a Boolean system.

Note that each nilpotent t-norm on $\mathbb{I}^{[2]}$ determines a unique weak Boolean system on $\mathbb{I}^{[2]}$. Note also that a Stone system on $\mathbb{I}^{[2]}$ satisfying either of the two conditions $x^* \bigtriangledown y^* = (x \triangle y)^*$ or $(x \triangle y)^* = (x \bigtriangledown y)^*$ satisfies both and thus is a weak Boolean system.

Let $\left(\mathbb{I}^{[2]}, \triangle, \bigtriangledown, *\right)$ be any Stone system on $\mathbb{I}^{[2]}$. The analogy with Stone algebras is apparent in the following observations that are reminiscent of the triple construction (see [17]). The image of $*$ is the sublattice $\{(c,c) : c \in [0,1]\}$ which is isomorphic to $\mathbb{I}$ and is a Boolean system under the induced operations. The kernel of $*$ is the sublattice $\{(a,1) : a \in [0,1]\}$. Every element of $\mathbb{I}^{[2]}$ is of the form $(a,b) = (c,1) \triangle (b,b)$ since $0 \circ b \leq a \leq 1 \circ b$ implies $a = c \circ b$ for some c.

7 A mathematical setting for fuzzy logics

In the previous sections, we have mainly been concerned with determining which systems are isomorphic and which are not. If the specified mathematical structure of an object is all that is used in an application, then certainly isomorphic objects are interchangeable and should not influence the quality of the model. In fuzzy logic applications, one may see statements to the effect that one model works better than another despite the fact that they are mathematically isomorphic. If these statements are true in a global sense then one can conclude that additional structure of the object not reflected in the mathematical structure, is being used. Thus a main point of our work is to identify which are the mathematically distinct choices of truth value algebras in fuzzy logic. In this section we go a step further. Even non-isomorphic algebras may lead to the same propositional logic.

We review the setup of a mathematical propositional logic in algebraic terms, describing when two choices of truth value algebras give the same logic. This analysis applies to the plethora of choices used in fuzzy systems. In particular we consider the propositional logic obtained when the algebra of truth values is $\mathbb{I}$, the real numbers in the unit interval equipped with minimum, maximum and $\neg x = 1 - x$ for conjunction, disjunction and negation, respectively. In this case we have propositional fuzzy logic, and we show that this is the same as three-valued logic. We also consider the case where the algebra of truth values is $\mathbb{I}^{[2]}$, the set $\{(a,b) \mid a \leq b \text{ and } a,b \in [0,1]\}$ with component-wise operations. This is what is sometimes called interval-valued fuzzy logic. We show that the logic obtained with the algebra of truth values $\mathbb{I}^{[2]}$ is the same as the one given by a certain four element lattice of truth values. Since both of these logics are equivalent to ones given by finite algebras, it follows that there are finite algorithms for determining when two statements are logically equivalent within either of these logics. On this topic, we discuss normal forms for both of these logics. To illustrate the use of these methods in the more general setting of t-norms, we show that the conjunctive logic obtained by using the unit interval with a strict Archimedean t-norm as the truth value algebra, is the same as the logic obtained by using the unit interval with a nilpotent Archimedean t-norm.

The general setup as well as the specific algebraic results exposed and applied in this section are well-known in universal algebra and logic, but it seems that many of the points that become transparent with this viewpoint are not well-known in fuzzy logic. A more detailed account of this work may be found in [20].

7.1 Propositional logic

Propositional logic deals with the properties of some set of logical connectives, most often the connectives 'and' ($\wedge$), 'or' ($\vee$), 'not' ($\neg$), and their derivates. These can be viewed as operations defined on sets of propositions. The connective 'and' for example would yield a binary operation on the set of propositions. So the first things that have to be specified are which connectives the logic to be built will deal with. This is done by giving a collection of connective symbols, such as $\wedge$, $\vee$, and $\neg$, and arities of the corresponding operations (in this case 2, 2, and 1). We then take a non-empty set X of variables, whose elements we think of as propositional variables, standing for propositions. Now the formulas of our propositional logic can be built up from the propositional variables by combining them with the connective symbols. The set $\mathcal{P}$ of all well-formed formulas is usually described inductively as the smallest subset of the set of all strings in the variables, connective symbols, and parentheses satisfying

1. If x is a propositional variable then x is a well-formed formula;
2. If α and β are well-formed formulas then so are $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\neg\alpha)$.

In $\mathcal{P}$ logical meaning is not accounted for; any two formulas that are distinct as strings of symbols are not equal as formulas. For example the strings $x \wedge y$ and $y \wedge x$ are different as strings even though they are identified in most logics. Thus it also is the case that the formulas that will carry the logic are the same for any two propositional logics of the same connective type. The difference and the logical meaning are introduced when an *algebra of truth values* is given. This is just an algebra $\mathbb{A}$ with the same number of basic operations as there are connective symbols and with matching arities. In many cases the operations corresponding to conjunction and disjunction are taken to be idempotent or to have other special properties. However, we wish to stress that we are not placing any such restrictions on the algebra $\mathbb{A}$.

An *interpretation* or *model* of the formulas is a map $t : X \to \mathbb{A}$, that is, a function that assigns a truth value to each propositional variable. This corresponds to interpreting the variables as specific propositions that have definite truth values. Once this has been done the truth value of every proposition corresponding to a well-formed formula can be determined. That is, we get a map $\tilde{t} : \mathcal{P} \to \mathbb{A}$ extending t simply by specifying that we want $\tilde{t}\,((\alpha \wedge \beta)) = \tilde{t}\,(\alpha) \wedge \tilde{t}\,(\beta)\,, \tilde{t}\,((\alpha \vee \beta)) = \tilde{t}\,(\alpha) \vee \tilde{t}\,(\beta)$, and $\tilde{t}\,((\neg\alpha)) = \neg\tilde{t}\,(\alpha)$ for all well-formed formulas α and β. This latter condition can be expressed by saying $\tilde{t}$ is a homomorphism between the two algebras. Here we consider $\mathcal{P}$ with the operations defined simply as concatenation. For example, given formulas α and β in $\mathcal{P}$, we define the operation on $\mathcal{P}$ corresponding to the operation symbol $\wedge$ to give the string $(\alpha \wedge \beta)$ in $\mathcal{P}$. Since $\mathcal{P}$ is generated by X as an algebra, knowing the action of $\tilde{t}$ on X is enough to completely determine $\tilde{t}$. Finally, the fact that every map $t : X \to \mathbb{A}$ gives rise to a homomorphism $\tilde{t} : \mathcal{P} \to \mathbb{A}$ reflects the fact that $\mathcal{P}$ is *freely generated* by the set X. Thus, we have a one-to-one correspondence between the set $\mathbb{A}^X$ of all interpretations of the variables and the set $Hom\,(\mathcal{P}, \mathbb{A})$ of all homomorphisms from $\mathcal{P}$ to $\mathbb{A}$.

These interpretations are what specify the propositional logic. In the propositional logic determined by $\mathbb{A}$, we say that two formulas p and q are *logically equivalent*, and we write $p \sim_{\mathbb{A}} q$, if for each interpretation, the truth value assigned to p is equal to the one assigned to q. The propositional logic $L_{\mathbb{A}}$ determined by $\mathbb{A}$ is the quotient algebra $\mathcal{P}/\sim_{\mathbb{A}}$ in which formulas from $\mathcal{P}$ are identified if they are logically equivalent.

Now if the propositional logic $L_{\mathbb{A}}$ determined by $\mathbb{A}$ is what we are interested in, then we should not distinguish between two algebras that give rise to the same logic. That is, the question becomes: when is $\mathcal{P}/\sim_{\mathbb{A}}= \mathcal{P}/\sim_{\mathbb{B}}$ for two algebras $\mathbb{A}$ and $\mathbb{B}$ of the same operational type? This question is a central point of universal algebra which provides powerful tools for answering this question.

7.2 Universal algebra for propositional logic

In universal algebraic terms, the set of connective symbols, or algebraically, operation symbols, together with their prescribed arities is called a **type**. A set equipped with one operation for each symbol in the type, such as the truth value algebra $\mathbb{A}$ of the appropriate arity is then called an **algebra** of that type.

In universal algebra, the set $\mathcal{P}$ of well-formed formulas is called the **term algebra** of that type. The algebraic importance of the elements of $\mathcal{P}$ is that they are the basic building block of algebraic equations: an equation of type τ is $p \approx q$ where $p, q \in \mathcal{P}$. Also $p \approx q$ holds in an algebra $\mathbb{A}$ if and only if for all interpretations $t : X \to \mathbb{A}$ of the variables in $\mathbb{A}$, $\tilde{t}\,(p) = \tilde{t}\,(q)$. That is, the logical equivalence relation $\sim_{\mathbb{A}}$ is exactly the equational theory of $\mathbb{A}$.

Now the algebra $L_{\mathbb{A}} = \mathcal{P}/\sim_{\mathbb{A}}$ is what is known as the free algebra over X given by $\mathbb{A}$, and a deep theorem of Birkhoff's tells us that $L_{\mathbb{A}} = L_{\mathbb{B}}$ if and only if $\mathcal{V}\,(\mathbb{A}) = \mathcal{V}\,(\mathbb{B})$ where $\mathcal{V}\,(\mathbb{A})$ is the least class of algebras containing $\mathbb{A}$ closed under homomorphic images, subalgebras, and products. Equivalently, it is the class of all algebras satisfying the equations that hold in $\mathbb{A}$.

Example 97 *Classical propositional logic is obtained when we choose $\mathbb{A}$ to be the two element Boolean algebra $\mathbf{2} = (\{0,1\}, \wedge, \vee, \neg, 0, 1)$[1]. In this case $L_2(X)$ is the free Boolean algebra freely generated by X. And we get the exact same propositional logic no matter which Boolean algebra we choose as the set of truth values, since any Boolean algebra generates the variety of all Boolean algebras.*

Other examples are provided by fuzzy logic and other non-classical logics presented in the next section.

If $\mathbb{A}$ is any finite algebra then there is a finite process for checking whether or not $\alpha \sim_{\mathbb{A}} \beta$, whence the logical equivalence relation $\sim_{\mathbb{A}}$ of the logic generated by $\mathbb{A}$ is given by a finite algorithm if there is a finite algebra $\mathbb{B}$ with $\mathcal{V}ar(\mathbb{A}) = \mathcal{V}ar(\mathbb{B})$.

Before closing this section we want to stress the consequences we can draw from this approach. When looking for an appropriate propositional logic structure for a particular situation, one just needs to study the universal algebraic properties of the

[1] We will use the same label for the operation in a specific algebra as for the corresponding function symbol, assuming that it will be clear from the context which we are talking about.

corresponding algebras of truth values. If it has already been decided that the algebra of truth values is to be in a certain variety for example, then there will be as many choices of distinct logics as there are subvarieties of that variety. A common choice of truth value structure in fuzzy logic is a deMorgan system (see [18]). In previous sections we determined which deMorgan systems on $\mathbb{I}$ and on $\mathbb{I}^{[2]}$ are isomorphic. Of course isomorphic algebras generate the same variety, but non-isomorphic algebras can generate the same variety also, and hence determine the same logic. Any two Boolean algebras generate the same variety, for example, and the same propositional logic. This is also illustrated by examples in the next section and by the following example from which we can conclude that all convex Archimedean t-norms give rise to the same logic.

Example 98 *All algebras* $(\mathbb{L}, \circ)$ *with* $\mathbb{L} = ([0,1], \wedge, \vee, 0, 1)$ *and* $\circ$ *a strict (continuous, Archimedean) t-norm are isomorphic to the algebra* $\mathbb{A} = (\mathbb{L}, \cdot)$ *where* $\cdot$ *is multiplication on the unit interval. And all algebras* $(\mathbb{L}, \circ)$ *with* $\circ$ *a nilpotent (continuous, Archimedean) t-norm are isomorphic to the system* $\mathbb{B} = (\mathbb{L}, \blacktriangle)$ *where* $x \blacktriangle y = (x + y - 1) \vee 0$*. The two algebras* $\mathbb{A} = (\mathbb{L}, \cdot)$ *and* $\mathbb{B} = (\mathbb{L}, \blacktriangle)$ *are not isomorphic. Let* $\mathcal{V}(\mathbb{A})$ *be the variety generated by* $\mathbb{A}$*,* $\mathcal{V}(\mathbb{B})$ *the variety generated by* $\mathbb{B}$*, and let* $a \in (0,1)$*. The relation* $\sim$ *on* $\mathbb{A}$ *given by* $x \sim y$ *if* $x, y \in [0, a]$ *is a congruence, and the quotient algebra* $\overline{\mathbb{A}} = \mathbb{A}/\sim$ *satisfies*

$$x \circ y = \begin{cases} x \cdot y & \text{if } x \cdot y > a \\ \bar{0} = \bar{a} & \text{if } x \cdot y \leq a \end{cases}$$

There are numbers $x, y > a$ *with* $x \cdot y \leq a$*, so this t-norm is nilpotent and* $\overline{\mathbb{A}}$ *is isomorphic to* $\mathbb{B}$*. Since varieties are closed under quotients, we see that* $\mathcal{V}(\mathbb{B}) \subseteq \mathcal{V}(\mathbb{A})$*.*

Now we find a subalgebra of $\prod_{\mathbb{Z}}^{+} \mathbb{B}$ *of the form* $(\mathbb{L}, \circ)$ *with* $\circ$ *a strict t-norm. For* n*,* m *positive integers,* r *a positive real number, and* $y \in (0,1)$*, the powers* $y^{[n]}$*,* $y^{\left[\frac{1}{m}\right]}$*, and* $y^{[r]}$ *are defined by*

$$y^{[n]} = \overbrace{y * y * \cdots * y}^{n \text{ times}} \qquad \left(y^{\left[\frac{1}{m}\right]}\right)^{[m]} = y \qquad y^{[r]} = \lim_{\frac{m}{n} \to r} \left(y^{\left[\frac{1}{m}\right]}\right)^{[n]}$$

Let $x_n \in \mathbb{B}$ *with* $x_n^{[n]} \neq 0$*,* $n \in \mathbb{Z}^{+}$*. Set* $a = (x_n) \in \prod_{\mathbb{Z}}^{+} \mathbb{B}$ *and set* $S = \left\{a^{[-\ln x]} : x \in [0,1]\right\}$*. Then* $\mathbb{S} = (S, \wedge, \vee, 0, 1, \circ)$*, where* $\wedge, \vee, 0, 1, \circ$ *are the operations inherited from the coordinatewise operations on* $\prod_{\mathbb{Z}^{+}} \mathbb{B}$*, is a subalgebra of* $\prod_{\mathbb{Z}^{+}} \mathbb{B}$ *isomorphic to* $\mathbb{A}$*. Since varieties are closed under both products and subalgebras, we see that* $\mathcal{V}(\mathbb{B}) = \mathcal{V}(\mathbb{A})$*.*

In contrast, we will show in future work that deMorgan systems with strict t-norms give rise to the same logic if and only if they are isomorphic!

7.3 Some fuzzy logics

The propositional logic underlying what people loosely call fuzzy logic, or Lee-Chang fuzzy logic [28], is the logic you get when you take $\mathbb{I}$ to be the algebra of truth values. Here $\mathbb{I} = ([0,1], \wedge, \vee, 0, 1, \neg)$ is the unit interval of real numbers with the lattice operations induced by the natural order, and $\neg x = 1 - x$ as the negation. The papers [35, 13] address the properties of the underlying propositional logic. The main question motivating both papers seems to be the existence of a finite algorithm for determining logical equivalence.

Elkan essentially keeps the same algebra of truth values (the only difference is that he does not retain the constants 0 and 1 in the type). However, he changes the class of truth valuations or interpretations, thus deviating from the natural construction of a propositional logic. In effect, he is just dealing with the natural classical two-valued logic. As far as we can tell, the reason for this strange choice of admissible interpretations is to have a logical equivalence that can be checked with a finite algorithm. But since this collapses to classical propositional logic, it is of no interest for fuzzy logic considerations.

In [35], the authors stay within the natural framework of propositional logic as described here, but they change the algebra of truth values to the algebra of subintervals. For this logic, which they call *applied fuzzy propositional logic*, they develop a normal form and thus a formal way of checking logical equivalence. The normal form they arrive at is the normal form for deMorgan algebras (with the bounds left out). In the process of developing the normal form, they show that it is the appropriate normal form both for their applied fuzzy logic and for logic programming. Whereas the normal form and its relation to the logic they propose is essentially well-known universal algebraic information, the fact that the logic obtained from interval-valued fuzzy logic (applied fuzzy logic) and the logic obtained from logic programming coincide seems to be a very interesting result.

The main justification provided in [35] for the switch to interval-valued was that this allows for a finite algorithm for checking logical equivalence. However, this is not a reasonable justification as there is a finite algorithm for checking logical equivalence in 'classical' fuzzy logic also. This has been known for a long time [32] and we will show it here as a direct consequence of well-known algebraic facts. According to the discussion in the last section, we just need to show the existence of a finite algebra generating the same variety. To this end we situate the algebras in question within algebraic theory.

The algebras in question are *deMorgan algebras:*

Definition 99 *A* deMorgan algebra *is an algebra* $\mathbb{L}$ *of the type* $(2,2,0,0,1)$ *satisfying*

1. $(L, \vee, \wedge, 0, 1)$ *is a distributive lattice with* $0, 1$*.*
2. $\neg(x \vee y) = \neg x \wedge \neg y$ *and* $\neg(x \wedge y) = \neg x \vee \neg y$ *are identities.*

3. $\neg(\neg x) = x$ *is an identity.*

We denote the equational class of deMorgan algebras by $\mathcal{M}$. A unary operation on a distributive lattice satisfying 2) and 3) is called an *involution*.

Example 100 *Let* **3** *denote the three element chain* $\{0, u, 1\}$ *with its unique involution. When we choose* $\mathbb{A}$ *to be the deMorgan algebra* **3** *then we get what is known as three-valued logic.*

The algebra $\mathbb{I} = ([0,1], \vee, \wedge, \neg, 0, 1)$ *as described above forms a deMorgan algebra. Using* $\mathbb{I}$ *as the algebra of truth values we get the propositional logic known as (classical) fuzzy logic.*

Given any deMorgan algebra $\mathbb{L}$*, let* $\mathbb{L}^{[2]} = (L^{[2]}, \vee, \wedge, \neg, 0, 1)$ *where* $L^{[2]} = \{(x,y) : x, y \in L \text{ and } x \leq y\}$*,* $\vee$ *and* $\wedge$ *are defined coordinate-wise,* $\neg(x,y) = (\neg y, \neg x)$*,* $0_{L^{[2]}} = (0_L, 0_L)$ *and* $1_{L^{[2]}} = (1_L, 1_L)$*. Then* $\mathbb{L}^{[2]}$ *is again a deMorgan algebra. Since the pairs* (x,y) *satisfy* $x \leq y$*, they can be thought of as subintervals of* L*. Indeed the deMorgan algebra* $\mathbb{I}^{[2]}$ *is the one used as the algebra of truth values in the propositional logic known as interval fuzzy logic or practical fuzzy logic [35].*

Which logic is generated by each of the above choices of truth value algebra is completely determined by which variety of deMorgan algebras is generated by the truth value algebra in question. Therefore it is of interest to know the subvarieties, that is, the equational subclasses of the class of deMorgan algebras. This has long since been worked out and it turns out that there are very few subvarieties.

We denote the variety of deMorgan algebras by $\mathcal{M}$; the trivial subvariety of $\mathcal{M}$, consisting of all one-element algebras, by $\mathcal{M}_{-1}$; the subvariety of $\mathcal{M}$ generated by **2**, **3**, and $\diamond$, by $\mathcal{M}_0$, $\mathcal{M}_1$, and $\mathcal{M}_2$, respectively, where

$$\diamond \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & / & & \backslash & \\ v & & & & w \\ & \backslash & & / & \\ & & 0 & & \end{array}$$

with $\neg v = v$, and $\neg w = w$.

Theorem 101 *[24] The subvarieties of* $\mathcal{M}$ *are*

$$\mathcal{M}_{-1} \subseteq \mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 = \mathcal{M}$$

Moreover, for $L \in \mathcal{M}$*,* $L \in \mathcal{M}_0$ *if and only if* $x \wedge \neg x = 0$ *is an identity in* L *(i.e.* $\mathcal{M}_0$ *is the class of Boolean algebras), and* $L \in \mathcal{M}_1$ *if and only if*

$$[(x \vee \neg x) \wedge (y \wedge \neg y)] \vee [(y \vee \neg y) \wedge (x \wedge \neg x)] = (y \wedge \neg y) \vee (x \wedge \neg x)$$

is an identity in L*.*

The latter equality is equivalent to the inequality

$$x \wedge \neg x \leq y \vee \neg y$$

[24] and consequently to the equality

$$(x \wedge \neg x) \wedge (y \vee \neg y) = (x \wedge \neg x)\,.$$

A distributive lattice in which the inequality $x \wedge \neg x \leq y \vee \neg y$ holds is called a *normal i-lattice* [24] or a *Kleene algebra* [8, 25].

Theorem 102 *The deMorgan algebras* $\mathbf{3}$ *and* $\mathbb{I}$ *both generate the subvariety* $\mathcal{M}_1$ *of* $\mathcal{M}$*.*

Proof. We know that $\mathbf{3}$ generates $\mathcal{M}_1$ by the definition of $\mathcal{M}_1$. Also, since $\mathbb{I}$ satisfies the equation $(x \wedge \neg x) \wedge (y \vee \neg y) = (x \wedge \neg x)$ we have $\mathcal{V}ar(\mathbb{I}) \subseteq \mathcal{M}_1$. On the other hand, $\mathbb{I}$ is not Boolean so $\mathcal{M}_0$ is a proper subvariety of $\mathcal{V}ar(\mathbb{I})$. It follows that $\mathcal{V}ar(\mathbb{I}) = \mathcal{M}_1$. ■

Theorem 103 *The deMorgan algebras* $\mathbf{3}^{[2]}$ *and* $\mathbb{I}^{[2]}$ *both generate the equational class* $\mathcal{M}$ *of all deMorgan algebras.*

Proof. We know that $\diamond$ generates $\mathcal{M}$. And it is easy to see that $\diamond$ can be realized as a subalgebra of $\mathbf{3}^{[2]}$ which can be realized as a subalgebra of $\mathbb{I}^{[2]}$. Thus $\mathcal{M} = \mathcal{V}ar(\diamond) \subseteq \mathcal{V}ar(\mathbf{3}^{[2]}) \subseteq \mathcal{V}ar(\mathbb{I}^{[2]})$. On the other hand, $\mathbb{I}^{[2]}$ is a deMorgan algebra, so $\mathcal{V}ar(\mathbb{I}^{[2]}) \subseteq \mathcal{M}$. It follows that $\mathcal{V}ar(\mathbf{3}^{[2]}) = \mathcal{V}ar(\mathbb{I}^{[2]}) = \mathcal{M}$. ■

Thus (classical) fuzzy propositional logic is the same as three-valued logic, and interval (practical) fuzzy propositional logic is the same as the four-valued logic with the algebra $\diamond$ of truth values. Now what import does this have for fuzzy set theory? Among other things, there is an algorithm for checking the equality of two expressions in fuzzy sets involving the connectives max, min, and the usual negation $x \to 1 - x$. We now turn to the related topic of normal forms.

7.4 Canonical forms

Even though the existence of an algorithm for checking logical equivalence, allows us to check whether two formulae are equivalent, it does not give a convenient, natural choice for which among equivalent expressions to work with. A normal form is a description of a canonical representative of each equivalence class and a process for reducing (or expanding as the case might be) an arbitrary expression to the normal form that is equivalent to it.

The normal form for the free Boolean algebra is of course well-known: every element is uniquely a disjunction of complete conjunctions of literals. Here a complete conjunction of literals is a conjunction of literals in which each variable occurs exactly once. The empty disjunction is 0, and the disjunction of all the complete conjunctions is 1.

The normal form for the free deMorgan algebra is also known and is described in detail in the paper [35] (except that they leave out the bounds): every element is uniquely an irredundant disjunction of conjunctions, each of which involves only literals (i.e., variables and their negations). Here we of course consider the order in which the conjunctions appear, as well as the order of the literals within each conjunction, as immaterial. The irredundancy lies in discarding any conjunction of literals involving fewer literals than another of the conjunctions present in the disjunction and in discarding repetitions of literals within each conjunction. Further, to incorporate the extremes, 0 and 1, we have to discard any conjunction involving 0 or $\neg 1$ (if no conjunctions are left, then the normal form for the element is 0); within each conjunction we must discard any occurrence of 1 and $\neg 0$ unless this literal makes up an entire conjunction (in which case the normal form for the entire formula is 1).

This leaves the middle case of the free Kleene algebra. The normal form for this algebra may also be well-known, but we were not able to find it in the literature. So we present here in slightly more detail than the two preceding cases a normal form for the free Kleene algebra.

The fact that is of fundamental use in finding all of these normal forms is that in each case (Boolean, Kleene, deMorgan) the free algebra generated by $x_1, x_2, \ldots, x_n$ is a bounded distributive lattice generated by the (finite) set of literals: x_1, x_2, …, x_n, $\neg x_1$, $\neg x_2$, …, $\neg x_n$. It is well known (and we will take this for granted here) that each element of a finite distributive lattice is the join of all the join irreducibles (i.e., elements that cannot be written as a proper join) below it. This is the content of Birkhoff's duality for finite distributive lattices. We will take advantage of a slight refinement of this: every element is the join of the maximal join irreducibles below that element.

Thus getting a normal form just boils down to determining the join irreducibles (and the ordering between them). It is clear that in each of the above cases, an element in the free algebra only stands a chance of being join irreducible if it is equal to a conjunction of literals or is equal to 1. The normal form for deMorgan algebras stems from realizing that all conjunctions of literals as well as 1, are join irreducible. The normal form for Boolean algebras stems from realizing that the only join irreducible elements in the Boolean case are the complete conjunctions of literals. In [20] we determine which elements are join irreducible in the Kleene case—namely, an element of the free Kleene algebra over the variables $X = \{x_1, x_2, \ldots, x_n\}$ is join irreducible if and only if it is equal to 1 or it is a conjunction of literals satisfying at least one of the following two conditions:

1. It contains at most one of the literals for each variable.

2. It contains at least one of the literals for each variable.

The procedure for putting an arbitrary word in $\mathcal{T}(x_1, ..., x_n)$ in Kleene normal form:

1. Given a formula (or word) w in $\mathcal{T}(x_1, x_2, \ldots, x_n)$, first use deMorgan's laws to move all the negations in, so that the word is rewritten as a word w_1 which is of lattice type in the literals, 0, and 1.

2. Next use the distributive law to obtain a new word w_2 from w_1 which is a disjunction of conjunctions involving the literals, 0, and 1. Given a word in lattice-disjunctive normal form, such as w_2, we will call the conjunctions that the word is written in, a disjunction of 'full' conjunctions of that word. At this point, discard any full conjunction in which 0 or $\neg 1$ appears as one of the conjuncts. Also discard any repetition of literals from any full conjunction, as well as 1 and $\neg 0$ from any full conjunction in which they do not appear alone (if a full conjunction consists entirely of 1's and $\neg 0$'s, then replace the whole thing by 1). This yields a word w_3.

3. Now discard all non-maximal conjunctions among the full conjunctions that w_3 is a disjunction of. The type of conjunctions we now are dealing with are either conjunctions of literals or 1 by itself. Of course 1 is above all the others. It is the case that one conjunction of literals is below another if and only if the former contains all the literals contained in the latter. This process yields a word w_4.

4. At this point, replace any full conjunction of literals, c, which contains both literals for at least one variable by the disjunction of all the conjunctions of literals below c that contain exactly one of the literals for each variable not occurring in c.

5. Finally, again discard all non-maximal conjunctions among the full conjunctions that are left, and if no conjunctions are left, then replace the word by 0. The word thus obtained is now in the normal form described above.

Example 104 *The Kleene normal form for the two equivalent expressions*

$$\begin{aligned} w &= A \wedge ((\neg A \wedge B) \vee (\neg A \wedge \neg B) \vee (\neg A \wedge C)) \\ w' &= A \wedge \neg A \end{aligned}$$

is

$$\begin{aligned} w'' = \; & (A \wedge \neg A \wedge B \wedge C) \vee (A \wedge \neg A \wedge B \wedge \neg C) \\ & \vee (A \wedge \neg A \wedge \neg B \wedge C) \vee (A \wedge \neg A \wedge \neg B \wedge \neg C) \end{aligned}$$

See [20] for details of proofs and examples and additional discussion of the mathematics of fuzzy logics.

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