Some comments on fuzzy normal forms

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***Abstract*— In this paper, we examine and compare de Morgan-, Kleene-, and Boolean-disjunctive and conjunctive normal forms in fuzzy settings. This generalizes papers of Turksen on the subject of Boolean-normal forms.**

I. Introduction

The notion of a normal form, or canonical form, pervades mathematics. This is the identification of a unique selection from each equivalence class of an equivalence relation together with a method of finding that selection, given an arbitrary element of an equivalence class. A mundane example is writing a rational number in lowest form—that is, with numerator and denominator relatively prime. Different equivalence relations on matrices lead to some well-known examples such as the rational and Jordan canonical forms. A normal form allows us, in particular, to check whether two expressions are equivalent.

We are interested in normal forms for algebras that give rise to classical and fuzzy logics, including Boolean, Kleene, and De Morgan algebras. Since the propositional logic given by the algebra $\mathbb{A}$ of truth values over the set of variables X is equal to the free algebra generated by X in the variety generated by $\mathbb{A}$, a normal form for the logic is the same as a normal form for the free algebra. In the paper [3] we spell out in detail the setup of a mathematical propositional logic in algebraic terms, describing exactly when two choices of truth value algebras give the same logic. This analysis applies to the plethora of choices used in fuzzy systems. In particular we consider the propositional logic obtained when the algebra of truth values is $\mathbb{I}$, the real numbers in the unit interval equipped with minimum, maximum and $\neg x = 1 - x$ for conjunction, disjunction and negation, respectively. In this case we have propositional fuzzy logic which we show is the same as three-valued logic. We also consider the case where the algebra of truth values is $\mathbb{I}^{[2]}$, the set $\{(a, b) \,|\, a \leq b \text{ and } a, b \in [0, 1]\}$ with component-wise operations. This is what is sometimes called interval-valued fuzzy logic. We show that the logic obtained with the algebra of truth values $\mathbb{I}^{[2]}$ is the same as the one given by a certain four element lattice of truth values. Since both of these logics are equivalent to ones given by finite algebras, it follows that there are finite algorithms for determining when two statements are logically equivalent within either of these logics. We discussed normal forms for both of these logics. We refer you to that paper [3] for the mathematical setting. The following presentation is a bit less formal.

When describing normal forms and procedures for obtaining them, we will work in the term algebra $\mathcal{T}(x_1, x_2, \ldots, x_n)$ of type $(2, 2, 1, 0, 0)$, with operation symbols $\wedge$ ("meet", "and", "intersection", "conjunction"), $\vee$ ("join", "or", "union", "disjunction"), $\neg$ ("not"), 0, 1, and variables $x_0, x_1, \ldots, x_n$. In the term algebra one can form polynomials in the usual manner using the variables, operation symbols, and parentheses. Such a polynomial will be called a **term**. Two terms in normal form are considered equal if the only difference between them occurs in the order and the association among the disjunctions or within the individual conjunctions (this makes sense since both join and meet are both commutative and associative). We will interpret these forms in the free De Morgan, Kleene, and Boolean algebras in the variables $x_1, x_2, \ldots, x_n$. These free algebras are quotients of the term algebra.

In this paper, all lattices will be **distributive**, meaning for all x, y, z in the lattice,

$$\begin{aligned} x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\ x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \end{aligned}$$

An element x of a lattice $(L, \wedge, \vee)$ is **join-irreducible** if $x = \vee_{y \in S} y$ for some set $S \subseteq L$ implies $x \in S$, and **meet-irreducible** if $x = \wedge_{y \in S} y$ for some set $S \subseteq L$ implies $x \in S$. The following is well known:

Every element of a finite distributive lattice has a unique irredundant representation as a join of join-irreducible elements. Similarly, every element of a finite distributive lattice has a unique irredundant representation as a meet of meet-irreducible elements.

The variables $x_1, x_2, \ldots, x_n$ as well as their negations $\neg x_1, \neg x_2, \ldots, \neg x_n$ are called **literals**. The fact that is of fundamental use in finding all of these normal forms is that in each case (Boolean, Kleene, De Morgan) the free algebra generated by the variables $x_1, x_2, \ldots, x_n$ is a bounded distributive lattice generated by the (finite) set of literals: x_1, x_2, …, x_n, $\neg x_1$, $\neg x_2$, …, $\neg x_n$. Getting a normal form just boils down to determining the join irreducibles (and the ordering between them). It is clear that in each of the above cases, an element in the free algebra only stands a chance of being join irreducible if it is equal to a (possibly empty) conjunction of literals or is equal to 1.

A De Morgan algebra $\mathbb{A}$ has the form $\mathbb{A} = (L, \wedge, \vee, 0, 1, \neg)$ where 0 and 1 are the unique smallest and largest, respectively, elements of the distributive lattice $\mathbb{L} = (L, \wedge, \vee)$ and $\neg$ is an involution on $\mathbb{L}$. The lattice $\mathbb{L}$ is partially ordered by $x \leq y$ if and only if $x \wedge y = x$ (or equivalently, $x \vee y = y$). The involution $\neg$ is an order reversing bijection on $\mathbb{L}$: for all $x, y \in L$, $x \leq y$ implies $\neg x \geq \neg y$. Moreover, the **De Morgan laws** are satisfied:

$$\begin{aligned} \neg(x \wedge y) &= \neg x \vee \neg y \\ \neg(x \vee y) &= \neg x \wedge \neg y \end{aligned}$$

Kleene algebras are De Morgan algebras that also satisfy the inequality

$$x \wedge \neg x \leq y \vee \neg y$$

and Boolean algebras are De Morgan algebras that also satisfy

$$x \wedge \neg x = 0 \text{ and } x \vee \neg x = 1$$

In fact, all Boolean algebras are Kleene algebras, and all Kleene algebras are De Morgan algebras. These algebras are all of type $(2, 2, 1, 0, 0)$, being a set with two binary operations $\wedge$ and $\vee$, one unary operation $\neg$, and two nullary operations 0 and 1.

We will outline procedures for obtaining the disjunctive normal forms. The corresponding conjunctive normal forms can be found by reversing the roles of join and meet and of 0 and 1 in the discussions.

II. Normal forms for free Boolean algebras

Classical propositional logic is obtained when the algebra of truth values is the two-element Boolean algebra. This logic is a free Boolean algebra. The disjunctive normal form for the free Boolean algebra on variables $x_1, ..., x_n$ is of course well-known: every element is uniquely a disjunction of complete conjunctions of literals. Here a **complete conjunction** of literals is a conjunction of literals in which each variable occurs exactly once (for each i, either x_i or $\neg x_i$ occurs, and not both). The empty disjunction is 0, and the disjunction of all the complete conjunctions is 1. The normal form for Boolean algebras stems from realizing that the only join irreducible elements in the Boolean case are the complete conjunctions of literals. Here we of course consider the order in which the conjunctions appear, as well as the order of the literals within each conjunction, as immaterial. The irredundancy lies in discarding any conjunction of literals involving more literals than another of the conjunctions present in the disjunction and in discarding repetitions of literals within each conjunction.

When a term is a disjunction of conjunctions of literals, each conjunction bounded on either side by a disjunction symbol will be called a **full conjunction** of that term. For example, the term $w = (x_1 \wedge x_2) \vee (x_2 \wedge x_5 \wedge x_3) \vee (x_5 \wedge x_2) \vee (x_2 \wedge x_1)$ has four full conjunctions: $(x_1 \wedge x_2)$, $(x_2 \wedge x_5 \wedge x_3)$, $(x_5 \wedge x_2)$, and $(x_2 \wedge x_1)$. The conjunction $(x_2 \wedge x_1)$ is **redundant** in this term, since the equivalent conjunction $(x_1 \wedge x_2)$ appears elsewhere in the term. The term $(x_2 \wedge x_5 \wedge x_3)$ is **non-maximal** since the conjunction $(x_5 \wedge x_2)$ lies above it. (One conjunction of literals is below another if and only if the former contains all the literals contained in the latter, and of course 1 is above all the others.)

The procedure for putting an arbitrary term $w \in \mathcal{T}(x_1, x_2, \ldots, x_n)$ in Boolean-disjunctive normal form is roughly as follows:

1) First use De Morgan's laws to move all the negations in, so that the term is rewritten in lattice type in the literals, 0, and 1.
2) Next use the distributive law to obtain a disjunction of conjunctions involving the literals and 0 and 1. Discard any of the full conjunctions in which 0 or $\neg 1$ appears as one of the conjuncts, or in which both literals appear for at least one variable. Also discard any repetition of literals from any full conjunction, as well as 1 and $\neg 0$ from any full conjunction in which they do not appear alone (if a full conjunction consists entirely of 1's and $\neg 0$'s, then replace the whole conjunction by 1). The conjunctions remaining are either conjunctions of literals (not more than one for each variable) or 1 by itself.
3) Now discard all non-maximal conjunctions among the full conjunctions that the term is a disjunction of.
4) At this point, replace any full conjunction of literals, c, that lacks a literal for at least one variable by the disjunction of all the conjunctions of literals below c that contain exactly one of the literals for each variable. This yields a term that is a disjunction of complete conjunctions.
5) Finally, again discard all repetitions of conjunctions among the full conjunctions that are left, and if no conjunctions are left, then replace the term by 0.

The term thus obtained is now in Boolean-disjunctive normal form. Note that when interpreting these terms in a Boolean algebra, the element obtained at each step is equal to the starting element.

III. Normal forms for free De Morgan algebras

Interval-valued fuzzy propositional logic is obtained when the algebra of truth values is the family of subintervals of the unit interval. This algebra is a free De Morgan algebra. The normal form for free De Morgan algebras stems from realizing that *all* conjunctions of literals as well as 1, are join irreducible. The De Morgan-disjunctive normal form is described in detail in the paper [7] (except that they leave out the bounds): every element is uniquely an irredundant disjunction of conjunctions, each of which involves only literals (i.e., variables and their negations). To incorporate the extremes, 0 and 1, we have to discard any conjunction involving 0 or $\neg 1$ (if no conjunctions are left, then the normal form for the element is 0); within each conjunction we must discard any occurrence of 1 and $\neg 0$ unless this literal makes up an entire conjunction (in which case the normal form for the entire formula is 1).

Following is a procedure for putting an arbitrary term w in the variables $x_1, ..., x_n$ in De Morgan-disjunctive normal form:

1) Given a term in $\mathcal{T}(x_1, x_2, \ldots, x_n)$, first use De Morgan's laws to move all the negations in, so that the term is rewritten as a term w_1 which is of lattice type in the literals, 0, and 1.
2) Next use the distributive law to obtain a new term w_2 from w_1 which is a disjunction of conjunctions involving the literals, 0, and 1. At this point, discard any full conjunction in which 0 or $\neg 1$ appears as one of the conjuncts. Also discard any repetition of literals from any full conjunction, as well as 1 and $\neg 0$ from any full conjunction in which they do not appear alone (if a full conjunction consists entirely of 1's and $\neg 0$'s, then replace the whole conjunction by 1). If no conjunctions are left, then replace the term by 0.This yields a term w_3.
3) Now discard all non-maximal conjunctions among the full conjunctions that w_3 is a disjunction of. The type of conjunctions we now are dealing with are either conjunctions of literals or 1 by itself. This process yields a term w_4.

The term thus obtained is now in De Morgan-disjunctive normal form, and represents the same element as w when interpreted in a De Morgan algebra.

IV. Normal forms for free Kleene algebras

Classical fuzzy propositional logic is obtained when the algebra of truth values is the unit interval. This logic is a free Kleene algebra. The case of the free Kleene algebra in a finite number of variables is described in detail in [3], where we showed that an element in such an algebra is join irreducible if and only if it is equal to 1 or it is a conjunction of literals satisfying at least one of the following two conditions:

1) It contains at most one of the literals for each variable.
2) It contains at least one of the literals for each variable.

This yields the normal form for Kleene algebras: A term w is in **Kleene-disjunctive normal form** provided it is equal to the string 0 or 1 or a (non-empty) disjunction of literals satisfying each of the conjunctions of literals involved either contains at most one of the literals for each variable or contains at least one of the literals for each variable; the conjunctions of literals involved are pairwise incomparable; and there is no repetition of literals within any one of the conjunctions.

Following is a procedure for putting an arbitrary term $w \in \mathcal{T}(x_1, x_2, \ldots, x_n)$ in Kleene normal form:

1) Follow the steps in the previous section to put the term in De Morgan-disjunctive normal form w_4.
2) At this point, replace any full conjunction of literals, c, which contains both literals for at least one variable by the disjunction of all the conjunctions of literals below c that contain exactly one of the literals for each variable not occurring in c. This process yields a term w_5.
3) Finally, again discard all non-maximal conjunctions among the full conjunctions that are left, and if no conjunctions are left, then replace the term by 0. This process yields a term w_6.

The term thus obtained is now in Kleene-disjunctive normal form. Note that when interpreting these terms in a Kleene algebra, the element obtained at each step is equal to the original element w interpreted in that Kleene algebra.

V. Comparison of normal forms for Boolean, Kleene, and De Morgan algebras

The procedure for finding the Kleene disjunctive normal form started with finding the De Morgan disjunctive normal form. The next steps replaced full conjunctions by conjunctions lying below them, and removed redundancies, so that, in the free De Morgan algebra in the variables $x_1, x_2, \ldots, x_n$ we have $w = w_4 \geq w_5 = w_6$.

Although it is a little longer, we could have written the procedure for finding the Boolean disjunctive normal form as follows:

1) Follow the steps in the previous two sections to put a term w in Kleene-disjunctive normal form w_6.
2) At this point, discard any full conjunction of literals, c, which contains both literals for at least one variable; and replace any full conjunction of literals, c, that does not contain both literals for any variable by the disjunction of all the conjunctions of literals below c that contain exactly one of the literals for each variable not occurring in c. This process yields a term w_7.
3) Finally, again discard all non-maximal conjunctions among the full conjunctions that are left, and if no conjunctions are left, then replace the term by 0. This process yields a term w_8 in Boolean-disjunctive normal form.

Note that $w_6 \geq w_7 = w_8$.

For any term $w \in \mathcal{T}(x_1, x_2, \ldots, x_n)$, let $\mathbf{D}_{\mathcal{B}}(w)$ denote the Boolean-disjunctive normal form, $\mathbf{D}_{\mathcal{K}}(w)$ the Kleene-disjunctive normal form, and $\mathbf{D}_{\mathcal{M}}(w)$ the De Morgan-disjunctive normal form of that term. Let $\mathbf{C}_{\mathcal{B}}(w)$, $\mathbf{C}_{\mathcal{K}}(w)$, and $\mathbf{C}_{\mathcal{M}}(w)$ denote the corresponding conjunctive normal forms. The procedures for constructing the disjunctive normal forms, and the dual procedures for constructing the conjunctive normal forms, make it clear that interpreted as elements of a De Morgan algebra the following equalities and inequalities hold:

$$\begin{aligned} \mathbf{D}_{\mathcal{B}}(w) &\leq \mathbf{D}_{\mathcal{K}}(w) \leq \mathbf{D}_{\mathcal{M}}(w) \\ &= \mathbf{C}_{\mathcal{M}}(w) \leq \mathbf{C}_{\mathcal{K}}(w) \leq \mathbf{C}_{\mathcal{B}}(w) \end{aligned}$$

Interpreting these elements in a Kleene algebra yields

$$\begin{aligned} \mathbf{D}_{\mathcal{B}}(w) &\leq \mathbf{D}_{\mathcal{K}}(w) = \mathbf{D}_{\mathcal{M}}(w) \\ &= \mathbf{C}_{\mathcal{M}}(w) = \mathbf{C}_{\mathcal{K}}(w) \leq \mathbf{C}_{\mathcal{B}}(w) \end{aligned}$$

and interpreting these elements in a Boolean algebra yields

$$\begin{aligned} \mathbf{D}_{\mathcal{B}}(w) &= \mathbf{D}_{\mathcal{K}}(w) = \mathbf{D}_{\mathcal{M}}(w) \\ &= \mathbf{C}_{\mathcal{M}}(w) = \mathbf{C}_{\mathcal{K}}(w) = \mathbf{C}_{\mathcal{B}}(w) \end{aligned}$$

In [8], Turksen obtains one of these inequalities in a very special case, namely $\mathbf{D}_{\mathcal{B}}(w) \leq \mathbf{C}_{\mathcal{B}}(w)$ when interpreted as an element of a free Kleene algebra on two variables, or as he describes it in ([8], Theorem 1), "for all the sixteen binary compositions." We have shown that his inequalities and others hold in complete generality, for any finite number of variables.

Turksen generalizes his inequalities to the setting of t-norms and negations "for some cases of certain families of the conjugate pairs of operators (Γ, Σ) that are twice differentiable." In the last section of this paper we discuss inequalities in the setting of t-norms and negations.

VI. Truth Tables

Since classical fuzzy logic is the same as three-valued logic, the disjunctive and conjunctive normal forms for the fuzzy case can also be recovered from truth tables. The table of truth values for the two variable case is

x	y	$x \wedge y$	$x \vee y$	$\neg x$
0	0	0	0	1
0	u	0	u	1
0	1	0	1	1
u	0	0	u	u
u	u	u	u	u
u	1	u	1	u
1	0	0	1	0
1	u	u	1	0
1	1	1	1	0

corresponding to the three-element lattice

1
|
u
|
0

The disjunctive normal forms for the fuzzy case are gotten from a truth table as follows:

- For those rows that have value 1 in the column of the expression to be put into disjunctive normal form, form the conjunction of the variables with truth value equal to 1 with the negations of the variables with truth value equal to 0 (and leave out variables with truth value equal to u).
- For those rows that have value u in the column of the expression to be put into disjunctive normal form, form the conjunction of the variables with truth value equal to 1 with the negations of the variables with truth value equal to 0, and with both the variables and the negated variables of the variables with truth value equal to u.

The irredundant Kleene-disjunctive normal form for the expression is then obtained by discarding redundant conjunctions—that is, any conjunction that contains the same, and possibly more, literals as another conjunction in the term. Follow a dual procedure (reverse the roles of 0 and 1 and reverse the roles of $\wedge$ and $\vee$) to obtain Kleene-conjunctive normal forms.

Since interval-valued fuzzy logic is the same as a certain four-valued logic, the disjunctive and conjunctive normal forms for the interval-valued fuzzy case can also be recovered from truth tables. The table of truth values for the two variable case is

x	y	$x \wedge y$	$x \vee y$	$\neg x$
0	0	0	0	1
0	u	0	u	1
0	v	0	v	1
0	1	0	1	1
u	0	0	u	u
u	u	u	u	u
u	v	0	1	u
u	1	u	1	u
v	0	0	1	v
v	u	0	1	v
v	v	v	v	v
v	1	v	1	v
1	0	0	1	0
1	u	u	1	0
1	v	v	1	0
1	1	1	1	0

corresponding to the four-element lattice

1
/ \
u v
\ /
0

with negation as in the table above.

The disjunctive normal forms for the intervalued-valued fuzzy case are gotten from a truth table as follows:

- For those rows that have value 1 in the column of the expression to be put into disjunctive normal form, and truth values 0 or 1 and possibly u or v (but not both) for the variables, form the conjunction of the variables with truth value equal to 1 with the negations of the variables with truth value equal to 0 (and leave out variables with truth value equal to u or v). If the row has values all u's (or all v's) pick up $p = 1$.
- For those rows that have value 1 in the column of the expression to be put into disjunctive normal form, and both u and v occur as truth values for the variables, form two conjunctions: one using the variables with

truth value equal to 1 and the negations of the variables with truth value equal to 0 and using both the variables and the negated variables of the variables with truth value equal to u, the other using the variables with truth value equal to 1 and the negations of the variables with truth value equal to 0 and using both the variables and the negated variables of the variables with truth value equal to v.

- For those rows that have value u in the column of the expression to be put into disjunctive normal form, form the conjunction using the variables with truth value equal to 1, the negations of the variables with truth value equal to 0, and using both the variables and the negated variables of the variables with truth value equal to u (and leave out variables with truth value equal to v).
- For those rows that have value v in the column of the expression to be put into disjunctive normal form, form the conjunction using the variables with truth value equal to 1, the negations of the variables with truth value equal to 0, and using both the variables and the negated variables of the variables with truth value equal to v (and leave out variables with truth value equal to u).

The irredundant De Morgan-disjunctive normal form for the expression is then obtained by discarding redundant conjunctions—that is, any conjunction that contains the same, and possibly more, literals as another conjunction in the term.

VII. Normal forms with t-norms and negations

A **De Morgan system** is an algebra

$$([0,1], \wedge, \vee, \triangle, \bigtriangledown, \neg, 0, 1)$$

of type $(2,2,2,2,1,0,0)$ where $([0,1], \wedge, \vee, 0, 1)$ is the unit interval with the usual operations, $\neg$ is an antiautomorphism (“involution”, “negation”), $\triangle$ is a t-norm, and $\bigtriangledown$ is the dual t-conorm satisfying the De Morgan law $x \bigtriangledown y = \neg((\neg x) \triangle (\neg y))$. All t-norms we consider in this paper will be continuous and Archimedean—that is, either “strict” or “nilpotent”. In [1] we observed that every continuous, Archimedean De Morgan system is isomorphic to one in which the t-norm is either multiplication $x \triangle y = xy$ or the Łukaciewisz t-norm $x \triangle y = (x + y - 1) \vee 0$. In [4], we show that the logics given by t-norms with negations are fundamentally infinite in nature, so that normal forms and truth tables are not available. One can, however, look at forms analogous to the disjunctive and conjunctive normal forms that occur in Boolean, Kleene, or De Morgan algebras and ask when the inequalities analogous to those in Section V. hold for these forms.

Turksen has looked at the Boolean-disjunctive normal form of a term in two variables and the Boolean-conjunctive normal form of that term. He concluded that “for some cases of certain families”, when these forms are interpreted in a De Morgan system by replacing the meet symbol by the t-norm $\triangle$ and the join symbol by the dual t-conorm $\bigtriangledown$, the disjunctive normal form is contained in the conjunctive normal form. The key inequalities he used in his investigation are

$$\begin{array}{rcl} (a \bigtriangledown b) \triangle (a \bigtriangledown \neg b) & \geq & a \\ (a \triangle b) \bigtriangledown (a \triangle \neg b) & \leq & a \end{array}$$

When these inequalities hold, some of the inequalities related to the Boolean disjunctive and conjunctive forms will also hold. These two inequalities, when quantified as “for all $a, b \in [0,1]$”, are equivalent. So we need consider only one of them.

Using the De Morgan law, the second inequality can be rewritten in terms of the t-norm and negation:

$$\neg([\neg(a \triangle b)] \triangle [\neg(a \triangle (\neg b))]) \leq a$$

or equivalently,

$$[\neg(a \triangle b)] \triangle [\neg(a \triangle (\neg b))] \geq \neg a$$

Under an isomorphism of a strict system with multiplication and some negation η, this inequality is equivalent to

$$\eta(ab)\, \eta(a(\eta(b))) \geq \eta(a)$$

and under an isomorphism of a nilpotent system with the Lukasiewicz t-norm and some negation η, this inequality is equivalent to

$$\eta(a \blacktriangle b) \blacktriangle \eta(a \blacktriangle (\eta(b))) \geq \eta(a)$$

where $\blacktriangle$ denotes the Lukasiewicz t-norm. These inequalities are generally easier to work with in these forms. We have determined cases where the inequality holds and cases where it fails. It holds, for example, for Frank t-norms with parameter ≥ 1 (in particular it holds for multiplication) and fails for Frank t-norms with parameter between 0 and 1. Similar situations occur for the other families described below, where we list the four t-norms considered by Turksen and give the negation determining the isomorphic system together with either multiplication or the Lukasiewicz t-norm.

Examples

- The system determined by the Frank t-norm

$$x \bigtriangledown_{F_r} y = \log_r \left(1 + \frac{(r^x - 1)(r^y - 1)}{r - 1}\right), r > 0, r \neq 1$$

and the negation $\alpha(x) = 1 - x$ is isomorphic to the system determined by multiplication and the Sugeno negation

$$\sigma_r(x) = \frac{1 - x}{1 - (r-1)x}$$

This gives the inequality

$$\left(\frac{1 - xy}{1 - (r-1)xy}\right)\left(\frac{1 - x\frac{1-y}{1-(r-1)y}}{1 - (r-1)x\frac{1-y}{1-(r-1)y}}\right) \geq \frac{1 - x}{1 - (r-1)x}$$

for $x, y \in [0,1]$ which can be shown to hold for $r \geq 1$ and to fail for $0 < r < 1$.

- The system determined by the Schweizer and Sklar t-norm

$$x \bigtriangledown_{S_r} y = 1 - \left((1-x)^r + (1-y)^r - (1-x)^r (1-y)^r \right)^{\frac{1}{r}}$$

$r > 0$ and the negation $\alpha(x) = 1 - x$ is isomorphic to the system determined by multiplication and the negation

$$\eta(x) = \left(1 - \left(1 - (1 - x^r)^{\frac{1}{r}} \right)^r \right)^{\frac{1}{r}}$$

- The system determined by the Weber t-norm

$$x \bigtriangledown_{W_r} y = (r(x+y-1) - (r-1)xy) \vee 0, r > 0, r \neq 1$$

and the negation $\alpha(x) = 1 - x$ is isomorphic to the system determined by the Lukaciewisz t-norm and the negation

$$\eta(x) = \log_r \frac{r}{1 + r - r^{1-x}}$$

- The system determined by the Yager t-norm

$$x \bigtriangledown_{Y_r} y = \left(1 - \left((1-x)^r + (1-y)^r \right)^{\frac{1}{r}} \right) \vee 0, r > 0$$

and the negation $\alpha(x) = 1 - x$ is isomorphic to the system determined by the Lukaciewisz t-norm and the negation

$$\eta(x) = 1 - \left(1 - (1-x)^{\frac{1}{r}} \right)^r$$

For all of the above t-norms and negations, the inequality in question can be shown to fail for some values of r. The question of exactly for what values of the parameter does the inequality hold is still under investigation.

VIII. Summary

The fact that Boolean logic, classical fuzzy logic, and interval-valued fuzzy logic are the same as certain two-valued, three-valued, and four-valued logics makes it possible to realize disjunctive and conjunctive normal forms for the fuzzy logics analogous to the familiar ones used in Boolean logic, and to work with finite truth tables. The algebraic relationships between these logics makes the relationship between the disjunctive and conjunctive normal forms of the three types immediately clear in complete generality, namely that

$$\begin{aligned} \mathbf{D}_{\mathcal{B}}(w) &\leq \mathbf{D}_{\mathcal{K}}(w) \leq \mathbf{D}_{\mathcal{M}}(w) \\ &= \mathbf{C}_{\mathcal{M}}(w) \leq \mathbf{C}_{\mathcal{K}}(w) \leq \mathbf{C}_{\mathcal{B}}(w) \end{aligned}$$

The logics given by t-norms with negations are fundamentally infinite in nature, so normal forms and truth tables are not available. However, one can examine cases where inequalities hold analogous to those above.

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