De Morgan Systems on the Unit Interval[*]

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Abstract

Logical connectives on fuzzy sets arise from those on the unit interval. The basic theory of these connectives is cast in an algebraic spirit with an emphasis on equivalence between the various systems that arise. Special attention is given to De Morgan systems with strict Archimedean t-norms and strong negations. A typical result is that any De Morgan system with strict t-norm and strong negation is isomorphic to one whose t-norm is multiplication.

1 Introduction

A **fuzzy subset** A of a set S is a mapping $A : S \to [0,1]$. Operations on the set $\mathcal{F}(S)$ of all such fuzzy subsets of S come from operations on $[0,1]$. Standard ones are $\wedge, \vee$, and $'$ given by

$$\begin{aligned}(A \wedge B)(s) &= \min\{A(s), B(s)\} \\ (A \vee B)(s) &= \max\{A(s), B(s)\} \\ A'(s) &= 1 - A(s)\end{aligned}$$

These (logical) connectives are usually referred to as "and", "or", and "not", or "intersection", "union", and "complement". The original theory of fuzzy sets was formulated using these operations, and they still play a fundamental role in both theory and applications. Viewing subsets of S as mappings $S \to \{0,1\}$, these operations clearly generalize the usual notions of intersection, union, and complement. But there are many other such generalizations and a huge literature dealing with them.

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There are extensive bibliographies in [4, 5, 7], for example. Our main concern will be with special classes of these connectives, particularly strict Archimedean t-norms and t-conorms, and strong negations. Such operators have generators. We give the notion of generators a new emphasis, and investigate systematically the isomorphisms between De Morgan systems on the unit interval.

2 The unit interval

Operations on fuzzy sets come from those on the unit interval $[0,1]$. This interval is endowed with an order $\leq$, and has many other mathematical features. But the system $\mathbb{I} = ([0,1], \leq)$ is the basic building block of the theory. Our concern will be with putting additional operations on it, namely t-norms and the like. But first we need some discussion of automorphisms and anti-automorphisms of $\mathbb{I}$.

Definition 1 *An* ***automorphism*** *of* $\mathbb{I}$ *is a one-to-one mapping* f *of* $\mathbb{I}$ *onto* $\mathbb{I}$ *such that* $f(a) \leq f(b)$ *if and only if* $a \leq b$. *An* ***anti-automorphism*** *of* $\mathbb{I}$ *is a one-to-one mapping* g *of* $\mathbb{I}$ *onto* $\mathbb{I}$ *such that* $g(a) \geq g(b)$ *if and only if* $a \leq b$.

Thus automorphisms preserve $\leq$ and anti-automorphisms reverse $\leq$. It should be noted that $f(0) = 0$ and $f(1) = 1$ for automorphisms f and $g(0) = 1$ and $g(1) = 0$ for anti-automorphisms g. Since discontinuities of monotone functions are jumps, these automorphisms and anti-automorphisms are continuous. Of course, automorphisms are strictly increasing, and anti-automorphisms are strictly decreasing. These mappings are plentiful. Any continuous strictly increasing map connecting $(0,0)$ and $(1,1)$ in the plane is an automorphism of $\mathbb{I}$. Figure 1 shows a couple of pictures.

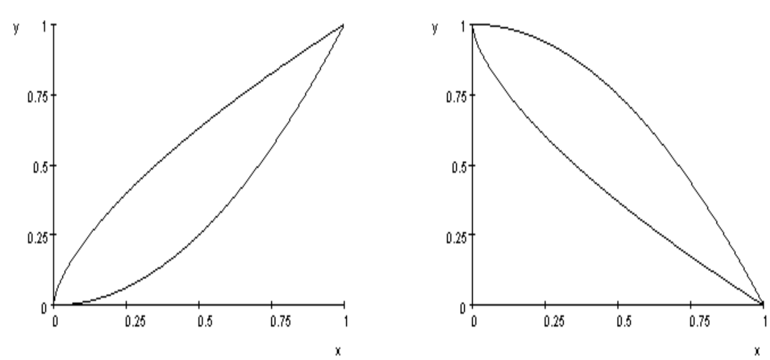


Figure 1. Automorphisms on the left, anti-automorphisms on the right

Let $Map(\mathbb{I})$ be the set consisting of all automorphisms and all anti-automorphisms of $\mathbb{I}$, and let $Aut(\mathbb{I})$ be the set of all automorphisms of $\mathbb{I}$. The elements of $Map(\mathbb{I})$ are functions, and may be composed. That is, if f and g are in $Map(\mathbb{I})$, fg is the element of $Map(\mathbb{I})$ given by $(fg)(x) = f(g(x))$. With this operation, $Map(\mathbb{I})$ is a

group. That is, composition of functions is a binary operation on $Map(\mathbb{I})$ that is associative, has an identity, and every element has an inverse. This means that

- $f(gh) = (fg)h$.
- There is an element 1 in $Map(\mathbb{I})$ such that $1f = f1 = f$ for all f. (The function 1 is the function given by $1(x) = x$ for all x. It is called the **identity** of the group.)
- For each $f \in Map(\mathbb{I})$, there is an element $f^{-1} \in Map(\mathbb{I})$ such that $ff^{-1} = f^{-1}f = 1$. (The element f^{-1} is simply the inverse of f as a function on $[0, 1]$.)

$Aut(\mathbb{I})$ is a **subgroup** of $Map(\mathbb{I})$. This means that the restriction of the operation on $Map(\mathbb{I})$ to $Aut(\mathbb{I})$ makes $Aut(\mathbb{I})$ into a group. This subgroup happens to be **normal**, that is, for every element f of $Aut(\mathbb{I})$ and g of $Map(\mathbb{I})$, the element $g^{-1}fg$ belongs to the subgroup $Aut(\mathbb{I})$. Elements of the form $g^{-1}fg$ are **conjugates** of f. Normal subgroups allow one to form the **quotient group** $Map(\mathbb{I})/Aut(\mathbb{I})$ whose elements are cosets $fAut(\mathbb{I}) = \{fg : g \in Aut(\mathbb{I})\}$ and with the operation $(fAut(\mathbb{I}))(gAut(\mathbb{I})) = fgAut(\mathbb{I})$. The normality of $Aut(\mathbb{I})$ makes this all work. The group $Map(\mathbb{I})/Aut(\mathbb{I})$ has only two elements, $Aut(\mathbb{I})$ and $fAut(\mathbb{I})$ for any $f \notin Aut(\mathbb{I})$.

There are a couple of other important subgroups. Each positive real number r gives an automorphism by $r(x) = x^r$. Identifying r with this automorphism, the set $\mathbb{R}^+$ of positive real numbers is a subgroup of $Aut(\mathbb{I})$. For any subset S of $Map(\mathbb{I})$ the set $\{x \in Map(\mathbb{I}) : xs = sx \text{ for all } s \in S\}$ is the **centralizer** $Z(S)$ of S in $Map(\mathbb{I})$ and is a subgroup of $Map(\mathbb{I})$. A particularly important anti-automorphism is α, given by $\alpha(x) = 1 - x$. Its centralizer consists of those $f \in Map(\mathbb{I})$ such that $f\alpha = \alpha f$. In this particular case, we will only be interested in those f which are in $Aut(\mathbb{I})$, that is, in the centralizer of α in $Aut(\mathbb{I})$, which is the group

$$Z(\{\alpha\}) \cap Aut(\mathbb{I}) = \{f \in Aut(\mathbb{I}) : f\alpha = \alpha f\}$$

For ease of notation, we are going to denote this group by $Z(\alpha)$, and more generally, for any $g \in Map(\mathbb{I})$,

$$Z(g) = \{f \in Aut(\mathbb{I}) : fg = gf\}$$

The group $Z(\alpha)$ consists exactly of those elements of $Aut(\mathbb{I})$ which commute with α, which is equivalent to

$$\begin{aligned} f\alpha(x) &= f(1-x) \\ &= \alpha f(x) \\ &= 1 - f(x) \end{aligned}$$

or that

$$f(x) + f(1-x) = 1.$$

Thus automorphisms that are in $Z(\alpha)$ look like those in Figure 2.

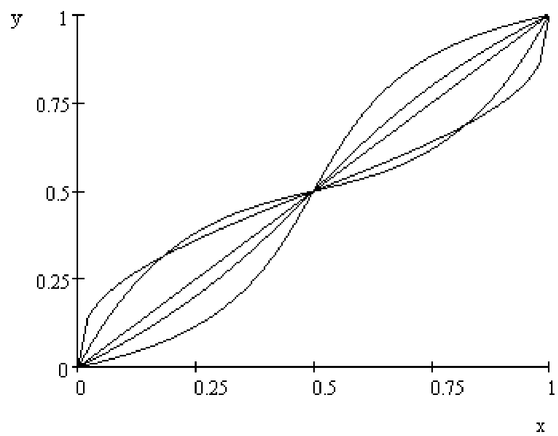


Figure 2. Some elements of $Z(\alpha)$.

Elements of $Z(\alpha)$ are easy to construct. For any $f \in Aut(\mathbb{I})$, the element $\frac{\alpha f \alpha + f}{2} \in Z(\alpha)$. In fact,

Theorem 2 $Z(\alpha) = \{\frac{\alpha f \alpha + f}{2} : f \in Aut(\mathbb{I})\}$.

Proof. If $f \in Z(\alpha)$, then $f = \frac{\alpha f \alpha + f}{2}$. For $f \in Aut(\mathbb{I})$,

$$\begin{aligned} \alpha\left(\left(\frac{\alpha f \alpha + f}{2}\right)(x)\right) &= 1 - \frac{1 - f(1-x) + f(x)}{2} \\ &= \frac{1 + f(1-x) - f(x)}{2} \\ &= \frac{(1 - f(x)) + f(1-x)}{2} \\ &= \left(\frac{\alpha f \alpha + f}{2}\right)\alpha(x) \end{aligned}$$

so $\frac{\alpha f \alpha + f}{2} \in Z(\alpha)$. ■

The map $\Phi : Aut(\mathbb{I}) \to Z(\alpha) : f \to \frac{\alpha f \alpha + f}{2}$ fixes $Z(\alpha)$ elementwise, but its group theoretical significance is not clear. For example, it is not a homomorphism: $\Phi(f)\Phi(g) = \Phi(f\Phi(g)) \neq \Phi(fg)$.

We will need the following later on. First, notice that for any $f \in Map(\mathbb{I})$ and any subgroup G of $Map(\mathbb{I})$,

$$f^{-1}Gf = \{f^{-1}gf : g \in G\}$$

is a subgroup of $Map(\mathbb{I})$.

Proposition 3 *For any f and $g \in Map(\mathbb{I})$,*

$$\left(f^{-1}\mathbb{R}^{+}f\right) \cap \left(g^{-1}Z(\alpha)g\right) = \{1\}$$

Proof. If $f^{-1}rf = g^{-1}zg$, then $gf^{-1}r = zgf^{-1}$. There is $x \in [0,1]$ such that $gf^{-1}(x) = \frac{1}{2}$. For this x, $gf^{-1}r(x) = zgf^{-1}(x) = z\left(\frac{1}{2}\right) = \frac{1}{2}$, and so $gf^{-1}(x^r) = \frac{1}{2}$. But $gf^{-1}(x) = \frac{1}{2}$, and since gf^{-1} is one-to-one, $r = 1$ and the proposition follows. ■

3 t-norms

We will put additional structure on the system $\mathbb{I}$, and first we consider t-norms. They are generalizations of intersection.

Definition 4 *A binary operation* $\circ$ *on* $[0,1]$ *is a* ***t-norm*** *if for all* $x, y, z \in [0,1]$,

1. $1 \circ x = x$
2. $x \circ y = y \circ x$
3. $(x \circ y) \circ z = x \circ (y \circ z)$
4. *The operation* $\circ$ *is increasing in each variable. That is,* $x \leq x_1$ *and* $y \leq y_1$ *imply that* $x \circ y \leq x_1 \circ y_1$.

Thus a binary operation on $[0,1]$ is a t-norm if 1 is an identity, it is commutative, associative, and increasing in each variable. A t-norm $\circ$ on $[0,1]$ gives a corresponding operation $\circ$ on fuzzy sets: $(A \circ B)(s) = A(s) \circ B(s)$. Of course, the associative property 3 gives unambiguous meaning to $x_1 \circ x_2 \circ ... \circ x_n$, and in particular to $x \circ x \circ ... \circ x$, which we write as x^n, where n is the number of x's. We have to be a little careful with this: x is a real number so for any real number r, x^r has meaning as the r-th power of x. The context will make clear the meaning of x^n. A t-norm $\circ$ has the following additional properties.

- $0 \circ x = 0$. This follows since $0 \circ x \leq 0 \circ 1 = 0$.
- $x \circ y = 1$ if and only if $x = y = 1$. Clearly $1 = 1 \circ 1$. If $1 = x \circ y$, then

$$1 = x \circ y \leq 1 \circ y = y$$

and similarly $1 = x$.

Two often used t-norms are min and ordinary multiplication. Many examples are given in the references.

Let $\circ$ be a t-norm and consider the system $(\mathbb{I}, \circ)$. This system is simply $\mathbb{I}$ with an additional structure on it, namely the operation $\circ$. Let $\diamond$ be another t-norm on $\mathbb{I}$. We need to make precise the notion of the systems $(\mathbb{I}, \circ)$ and $(\mathbb{I}, \diamond)$ being structurally the same.

Definition 5 *Let* $\circ$ *and* $\diamond$ *be t-norms. The systems* $(\mathbb{I}, \circ)$ *and* $(\mathbb{I}, \diamond)$ *are* ***isomorphic*** *if there is an element* $h \in Aut(\mathbb{I})$ *such that* $h(x \circ y) = h(x) \diamond h(y)$. *We write* $(\mathbb{I}, \circ) \approx (\mathbb{I}, \diamond)$. *The mapping* h *is an* ***isomorphism***. *The t-norms* $\circ$ *and* $\diamond$ *are* ***equivalent*** *if* $(\mathbb{I}, \circ) \approx (\mathbb{I}, \diamond)$.

This means that the systems $([0,1], \leq, \circ)$ and $([0,1], \leq, \diamond)$ are isomorphic in the sense of universal algebra: there is a one-to-one map from $[0,1]$ onto $[0,1]$ that preserves the operations and relations involved.

Equivalence between t-norms is an equivalence relation and partitions t-norms into equivalence classes. The t-norm min is rather special. A t-norm $\circ$ is **idempotent** if $a \circ a = a$ for all $a \in [0,1]$. If $\circ$ is idempotent, then for $a \leq b$, $a = a \circ a \leq a \circ b \leq a \circ 1 = a$, so $\circ = \min$. Thus min is the *only* idempotent t-norm. It is in an equivalence class all by itself. *In this article, we will restrict our attention to those t-norms $\circ$ such that $a \circ a < a$ for $a \in (0,1)$.*

An isomorphism of a system with itself is called an **automorphism**. It is easy to show that the set of automorphisms of $(\mathbb{I}, \diamond)$ is a subgroup of $Aut(\mathbb{I})$. Thus, with each t-norm $\diamond$, there is a group associated with it, namely its **automorphism group**

$$Aut(\mathbb{I}, \diamond) = \{f \in Aut(\mathbb{I}) : f(x \diamond y) = f(x) \diamond f(y)\}$$

For the t-norm $a \wedge b = \min\{a, b\}$, it is clear that $Aut(\mathbb{I}, \wedge) = Aut(\mathbb{I})$. For the t-norm multiplication, $Aut(\mathbb{I}, \cdot) = \{f \in Aut(\mathbb{I}) : f(x) = x^c, c \in \mathbb{R}^+\} \simeq \mathbb{R}^+$. This is a well-known, classical result related to one of the four basic functional equations of Cauchy (see, for example, Aczél [3]).

If H is a subgroup of a group G, and $g \in G$, then we have already noted that $g^{-1}Hg = \{g^{-1}hg : h \in H\}$ is a subgroup of G. This subgroup is said to be **conjugate to** H, or a **conjugate of** H. The map $h \to g^{-1}hg$ is an isomorphism from H to its conjugate $g^{-1}Hg$. By an **isomorphism** from a group G to a group H we mean a one-to-one onto map $\varphi : G \to H$ such that $\varphi(xy) = \varphi(x)\varphi(y)$.

Theorem 6 *If two t-norms are equivalent then their automorphism groups are conjugate.*

Proof. Suppose that $\circ$ and $\diamond$ are equivalent. Then there is an isomorphism $f : (\mathbb{I}, \circ) \to (\mathbb{I}, \diamond)$. The map $g \to f^{-1}gf$ is an isomorphism from $Aut(\mathbb{I}, \diamond)$ to $Aut(\mathbb{I}, \circ)$, so $f^{-1}Aut(\mathbb{I}, \diamond)f = Aut(\mathbb{I}, \circ)$. ■

Definition 7 *A t-norm $\circ$ is **convex** if whenever $x \circ y \leq c \leq x_1 \circ y_1$, then there is an r between x and x_1 and an s between y and y_1 such that $c = r \circ s$. A t-norm $\circ$ is **Archimedean** if for each $a, b \in (0,1)$, there is a positive integer n such that $a^n = \overbrace{a \circ a \circ \cdots \circ a}^{n \text{ times}} < b$.*

For t-norms, the condition of convexity is equivalent to continuity. In this article, we refer to the condition as convex. This formulation has the advantage of being strictly order theoretic, allowing us to remain within the algebraic context of $\mathbb{I}$ as a lattice. For convex t-norms, the condition for Archimedean simplifies, as the following well-known proposition attests.

Proposition 8 *The following are equivalent for a convex t-norm* $\circ$*.*

1. $\circ$ *isArchimedean.*

2. $a \circ a < a$ *for all* $a \in (0,1)$*.*

The theorem below is fundamental in determining equivalences of convex Archimedean t-norms. It has usually been thought of as a theorem about representing t-norms by generators. A principle reference is [8]. The theorem in essence goes back at least to Abel [1]. There is a proof for the strict case in [2], some discussion in [10], and a proof in [11]. We will give only a very brief outline of a proof here.

Theorem 9 *If* $\circ$ *is a convex Archimedean t-norm then there is an* $a \in [0,1)$ *and an isomorphism*

$$f : \mathbb{I} \to ([a,1], \leq)$$

such that

$$f(x \circ y) = \max\{f(x)f(y), a\}$$

for all $x, y \in [0,1]$*. Also if* $g : \mathbb{I} \to ([b,1], \leq)$ *is another such isomorphism, then* $g(x \circ y) = \max\{g(x)g(y), b\}$ *if and only if* $f = rg$ *for some* $r > 0$*.*

Proof. The proof consists of a construction of a map f satisfying $f(x \circ y) = f(x) \cdot f(y)$ for $x \circ y \neq 0$. An increasing sequence $\{x_n\}_{n=-\infty}^{\infty}$ in $[0,1)$ is defined inductively by the condition $x_n \circ x_n = x_{n-1}$. The set of all points of the form $x_{i_1} \circ x_{i_2} \circ \cdots \circ x_{i_n}$ is dense in the unit interval. Define a function f on the sequence $\{x_n : x_n \neq 0\}$ by

$$f(x_n) = x_0^{2^{-n}} \text{ if } x_n \neq 0$$

The function f can be extended to finite nonzero products under $\circ$ of the elements of this sequence by

$$f(x_{i_1} \circ x_{i_2} \circ \cdots \circ x_{i_n}) = f(x_{i_1}) f(x_{i_2}) \cdots f(x_{i_n}).$$

and then extended to an isomorphism from $\mathbb{I}$ to $([f(0), 1], \leq)$ by convexity. Then $f(x \circ y) = f(x)f(y)$ whenever $x \circ y \neq 0$, so that $x \circ y = f^{-1}(f(x)f(y))$ if $f(x)f(y) \geq f(0)$ and $x \circ y = 0$ otherwise.

Suppose that an isomorphism $g : I \to ([g(0), 1], \leq)$ gives the same t-norm as the f just constructed. Take $r = (\ln x_0) / (\ln g(x_0))$. Then $rg(x_0) = (g(x_0))^r = x_0 = f(x_0)$, and it can be seen from the construction of f that rg must agree with f on the points x_n and hence everywhere. Conversely, it is easy to show that if $f = rg$, then f and g give the same t-norm. ■

A function f such that $x \circ y = f^{-1}(\max\{f(x)f(y), f(0)\})$ is called a **generator** of the t-norm $\circ$. Two functions f and g are generators of the same t-norm if and only if $f = rg$ for some $r > 0$, that is, $f(x) = (g(x))^r$ for all $x \in [0,1]$.

A t-norm $\circ$ is **nilpotent** if for $a \neq 1$, $a^n = 0$ for some positive integer n, the n depending on a. It is clear from the theorem that $\circ$ is nilpotent if and only if $f(0) > 0$. So Archimedean t-norms fall naturally into two classes: nilpotent ones and those not nilpotent. Those not nilpotent are called **strict.** In the next section, we start by examining the strict ones.

Historically, Archimedean t-norms have been represented by maps $g : I \to [0, \infty]$, where g is a strictly decreasing function with $0 < g(0) \leq \infty$ and $g(1) = 0$. In this case the binary operation satisfies

$$g(x \circ_{g^+} y) = \min\{g(x) + g(y), g(0)\}$$

and since this minimum is in the range of g,

$$x \circ_{g^+} y = g^{-1}(\min\{g(x) + g(y), g(0)\})$$

These two types of representations give the same t-norms. In particular, if $g : I \to [0, \infty]$ is a continuous, strictly decreasing function, with $0 < g(0) \leq \infty$ and $g(1) = 0$, let $f(x) = e^{-g(x)}$, $x \circ_f y = f^{-1}(\max\{f(x)f(y), f(0)\})$ and $x \circ_g y = g^{-1}(\min\{g(x) + g(y), g(0)\})$. Then $f : \mathbb{I} \to [f(0), 1]$ is an isomorphism and $\circ_{g^+} = \circ_f$. (See [11], for example.) In this paper, we use the multiplicative representation, since this allows us to remain within the context of the unit interval and to have a natural group structure on the set of generators.

We restate the previous theorem for the strict t-norm case.

Theorem 10 *The Archimedean t-norm $\circ$ is strict if and only if there is an element $f \in Aut(\mathbb{I})$ such that $f(x \circ y) = f(x)f(y)$. Another element $g \in Aut(\mathbb{I})$ satisfies this condition if and only if $f = rg$ for some $r > 0$.*

So a generator of a strict t-norm $\circ$ is just an isomorphism from $Aut(\mathbb{I}, \circ)$ to $Aut(\mathbb{I}, \cdot)$.

Corollary 11 *For any strict t-norm $\circ$, $Aut(\mathbb{I}, \circ) \approx Aut(\mathbb{I}, \cdot)$.*

Corollary 12 *For any two strict t-norms $\circ$ and $\diamond$, $Aut(\mathbb{I}, \circ) \approx Aut(\mathbb{I}, \diamond)$.*

Additional properties of strict t-norms $\circ$ are these:

- On $(0, 1)$ the operation $\circ$ is strictly increasing in each variable. In fact, if $\circ$ is strict, then $f(x) = y \circ x$ is a one-to-one map of $[0, 1]$ onto $[0, y]$. This follows from the convexity and strict monotonicity of $\circ$.

- If $\circ$ is strict, then $f(x) = x \circ x$ is an automorphism and $g(x) = (1 - x) \circ (1 - x)$ is an anti-automorphism of $\mathbb{I}$. Again, this follows from the convexity and strict monotonicity of $\circ$.

- Multiplication is a strict t-norm.

Call two automorphisms f and g **equivalent** if they give the same strict t-norm, and write $f \sim g$. Then $\sim$ is an equivalence relation and so induces a partition of the group $Aut(\mathbb{I})$. The members of this partition are the **right cosets** $\{\mathbb{R}^+ f : f \in Aut(\mathbb{I})\}$ of $\mathbb{R}^+$. So the set of strict t-norms of $\mathbb{I}$ is in natural one-to-one correspondence with the right cosets in $Aut(\mathbb{I})$ of the subgroup $\mathbb{R}^+$. Rephrasing, we have

Corollary 13 *For an automorphism f of $\mathbb{I}$, let $\circ_f$ be given by $x \circ_f y = f^{-1}(f(x) f(y))$. Then*

$$\circ_f \to \mathbb{R}^+ f$$

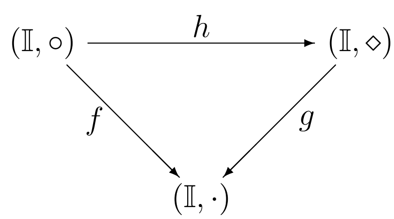
is a one-to-one correspondence between the strict t-norms on $[0,1]$ and the right cosets of the subgroup $\mathbb{R}^+$ in $Aut(\mathbb{I})$.

We know that for strict t-norms $\circ$ and $\diamond$, the systems $(\mathbb{I}, \circ)$ and $(\mathbb{I}, \diamond)$ are isomorphic. We spell out exactly what those isomorphisms are.

Theorem 14 *Let $\circ$ and $\diamond$ be strict t-norms with generators f and g, respectively. Then $h : (\mathbb{I}, \circ) \to (\mathbb{I}, \diamond)$ is an isomorphism if and only if $g^{-1} r f = h$ for some $r > 0$. That is, the set of isomorphisms from $(\mathbb{I}, \circ)$ to $(\mathbb{I}, \diamond)$ is the set*

$$g^{-1}\mathbb{R}^+ f = \{g^{-1} r f : r \in \mathbb{R}^+\}.$$

Proof. An isomorphism $h : (\mathbb{I}, \circ) \to (\mathbb{I}, \diamond)$ gets an isomorphism $(\mathbb{I}, \circ) \to (\mathbb{I}, \cdot)$ which must be rf for some $r \in \mathbb{R}^+$. So $h = g^{-1} r f$. For any r, $g^{-1} r f$ is an isomorphism.



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Corollary 15 *Let f be a generator of the strict t-norm $\circ$. Then*

$$Aut(\mathbb{I}, \circ) = f^{-1}\mathbb{R}^+ f \approx \mathbb{R}^+.$$

Proof. The set of automorphisms of $(\mathbb{I}, \circ)$ is $f^{-1}\mathbb{R}^+ f$. It is a subgroup of $Aut(\mathbb{I})$, and is isomorphic to $\mathbb{R}^+$ via the mapping $f^{-1} r f \to r$. ■

Corollary 16 $Aut(\mathbb{I}, \cdot) = \mathbb{R}^+$.

There are two basic facts about nilpotent (convex Archimedean) t-norms: any two are equivalent, and each has a trivial automorphism group.

Theorem 17 *Let $\circ$ and $\diamond$ be nilpotent t-norms with generators f and g respectively. Let $r \in \mathbb{R}^+$ with $g(0) = (f(0))^r$. Then $g^{-1}rf$ is the unique isomorphism from $(\mathbb{I}, \circ)$ to $(\mathbb{I}, \diamond)$.*

Proof. We may take $r = 1$, so that $f(0) = g(0)$. First note that $g^{-1}f \in Aut(\mathbb{I})$. We need to show that $g^{-1}f(a \circ b) = g^{-1}f(a) \diamond g^{-1}f(b)$, that is, that

$$g^{-1}ff^{-1}(\max\{f(a)f(b), f(0)\}) = g^{-1}(\max\{gg^{-1}f(a)gg^{-1}f(b), g(0)\})$$

which is clear since $f(0) = g(0)$. Suppose that $\varphi : (\mathbb{I}, \circ) \to (\mathbb{I}, \diamond)$ is an isomorphism. Then

$$\varphi f^{-1}(\max\{f(a)f(b), f(0)\}) = g^{-1}(\max\{(g\varphi)(a)(g\varphi)(b), g\varphi(0)\})$$

Thus

$$f^{-1}(\max\{f(a)f(b), f(0)\}) = \varphi^{-1}g^{-1}(\max\{(g\varphi)(a)(g\varphi)(b), g\varphi(0)\})$$

Since f and $g\varphi$ generate the same nilpotent t-norm and agree on 0, $f = g\varphi$ or $\varphi = g^{-1}f$ as asserted. ■

Corollary 18 *If $\circ$ is a nilpotent t-norm, then $Aut(\mathbb{I}, \circ) = \{1\}$.*

In the case of strict (convex Archimedean) t-norms we have that $Aut(\mathbb{I}, \circ) = f^{-1}R^+f \subseteq Aut(\mathbb{I})$. It turns out that these are the only convex t-norms with such automorphism groups.

Proposition 19 *Let $\circ$ be a convex t-norm. Then $Aut(\mathbb{I}, \circ) = f^{-1}\mathbb{R}^+f$ for some $f \in Aut(\mathbb{I})$ if and only if $\circ$ is a strict t-norm.*

Proof. Suppose that $Aut(\mathbb{I}, \circ) = f^{-1}\mathbb{R}^+f$ and suppose that for some $a \in (0, 1)$, $a \circ a = a$. Then for any $b \in (0, 1)$, there is an element $g \in Aut(\mathbb{I}, \circ)$ such that $g(a) = b$, namely $g = f^{-1}rf$ where $r = \ln f(b) / \ln f(a)$. Thus

$$b \circ b = g(a) \circ g(a) = g(a \circ a) = g(a) = b,$$

so that $\circ$ is idempotent. But the only idempotent t-norm is min, and $Aut(\mathbb{I}, \min) = Aut(\mathbb{I}) \neq f^{-1}\mathbb{R}^+f$. Thus $a \circ a < a$ for all $a \in (0, 1)$, and $\circ$ is Archimedean. The t-norm is not nilpotent, by the preceding corollary, and thus it is strict. ■

We cannot conclude that the function f in the proposition is a generator of $\circ$. Comment 3 at the end of this article refers to example of convex t-norms $\circ$ different from multiplication for which $Aut(\mathbb{I}, \circ) = \mathbb{R}^+$. In that case, $f = 1$ but 1 is clearly not a generator for $\circ$.

4 Negations

In this section, we will prove results analogous to those for t-norms in the previous sections. An element $f \in Map(\mathbb{I})$ has **order** n if $f^n = 1$, and n is the smallest such positive integer. If no such integer exists, the element has **infinite** order. All the elements of $Aut(\mathbb{I})$ have infinite order except 1 which has order 1. All anti-automorphisms are either of order two or of infinite order. Anti-automorphisms of order 2 are called **involutions**.

We will only concern ourselves with what are usually called strong negations. We will call them simply negations.

Definition 20 *A **negation** (or involution) on* $\mathbb{I}$ *is an anti-automorphism of* $\mathbb{I}$ *of order two.*

Negations will generally be denoted by small Greek letters. Thus a negation is an order-reversing, one-to-one mapping β of $\mathbb{I}$ onto $\mathbb{I}$ such that $\beta(\beta(x)) = x$, or equivalently such that $\beta^2 = 1$.

We reserve the notation α to denote the negation given by $\alpha(x) = 1 - x$. It is a trivial fact that conjugates of negations are negations, but it is a little less trivial and a bit surprising that every negation is a conjugate of α by an automorphism.

Theorem 21 *Let* β *be a negation in* $Map(\mathbb{I})$. *Let*

$$f(x) = \frac{1 - \beta(x) + x}{2}$$

Then $f \in Aut(\mathbb{I})$, *and* $\beta = f^{-1}\alpha f$. *Furthermore,* $g^{-1}\alpha g = \beta$ *if and only if* $gf^{-1} \in Z(\alpha)$.

Proof. Since f is the average of two automorphisms, it is an automorphism.

$$\begin{aligned} f\beta(x) &= \frac{1 - \beta(\beta(x) + \beta(x)}{2} \\ &= \frac{1 - x + \beta(x)}{2} \end{aligned}$$

and

$$\begin{aligned} \alpha f(x) &= 1 - \frac{1 - \beta(x) + x}{2} \\ &= \frac{1 + \beta(x) - x}{2} \end{aligned}$$

so we have $\beta = f^{-1}\alpha f$. Now, $g^{-1}\alpha g = f^{-1}\alpha f$ if and only if $gf^{-1}\alpha f g^{-1} = \alpha$ if and only if $gf^{-1}\alpha = \alpha g f^{-1}$ if and only if $gf^{-1} \in Z(\alpha)$. ■

We note that since $Z(f^{-1}\alpha f) = f^{-1}Z(\alpha)f$, the centralizer $Z(\beta)$ of a negation $\beta = f^{-1}\alpha f$ is the group

$$f^{-1}Z(\alpha)f = f^{-1}\left\{\frac{\alpha g\alpha + g}{2} : g \in Aut(\mathbb{I})\right\} f$$

An automorphism f such that $\beta = f^{-1}\alpha f$ is a **generator** of β. So every involution has a generator, and we know when two elements of $Aut(\mathbb{I})$ give the same involution. This theorem seems to be due to Trillas [12] who takes as generators functions from $[0,1]$ to $[0,\infty]$ [6]. A thrust of this paper is to use elements of $Map(\mathbb{I})$ as generators of t-norms, t-conorms, and negations. This enables us to use the language of group theory and to involve only functions on $[0,1]$. Also, we are automatically provided with natural operations between generators, being elements of the group $Map(\mathbb{I})$.

Theorem 22 *Let β be a negation and let f be a generator of β. The map $\beta \to Z(\alpha)f$ is a one-to-one correspondence between the negations of $\mathbb{I}$ and the set of right cosets of the centralizer $Z(\alpha)$ of α.*

Consider two systems $(\mathbb{I},\beta)$ and $(\mathbb{I},\gamma)$ where β and γ are negations. They are **isomorphic** if there is a map $h \in Aut(\mathbb{I})$ with $h(\beta(x)) = \gamma h(x)$, that is if $h\beta = \gamma h$, or equivalently if $\beta = h^{-1}\gamma h$. Let f and g be generators of β and γ, respectively. If h is an isomorphism, then $hf^{-1}\alpha f = g^{-1}\alpha gh$ which means that

$$f^{-1}\alpha f = h^{-1}g^{-1}\alpha gh = (gh)^{-1}\alpha gh$$

Therefore, f and gh generate the same negation, and so $zf = gh$ for some $z \in Z(\alpha)$. Thus $h \in g^{-1}Z(\alpha)f$. It is easy to check that elements of $g^{-1}Z(\alpha)f$ are isomorphisms $(\mathbb{I},\beta) \to (\mathbb{I},\gamma)$. We have the following theorem.

Theorem 23 *Let β and γ be negations with generators f and g, respectively. Then the set of isomorphisms from $(\mathbb{I},\beta)$ to $(\mathbb{I},\gamma)$ is $g^{-1}Z(\alpha)f$. In particular, $g^{-1}f$ is an isomorphism from $(\mathbb{I},\beta)$ to $(\mathbb{I},\gamma)$.*

Note that $Z(\alpha)$ plays a role for negations analogous to that of $\mathbb{R}^+$ for strict t-norms. See Corollary 13. If we call two negations β and γ **equivalent** if $(\mathbb{I},\beta) \approx (\mathbb{I},\gamma)$, then the previous theorem says in particular that any two negations are equivalent. We have the following special cases.

Corollary 24 *The set of isomorphisms from $(\mathbb{I},\beta)$ to $(\mathbb{I},\alpha)$ is the right coset $Z(\alpha)f$ of $Z(\alpha)$. In particular, the generator f of β is an isomorphism from $(\mathbb{I},\beta)$ to $(\mathbb{I},\alpha)$.*

Noting that $f^{-1}Z(\alpha)f = Z(\beta)$, we have

Corollary 25 *$Aut(\mathbb{I},\beta) = f^{-1}Z(\alpha)f = Z(\beta)$. In particular, $Aut(\mathbb{I},\alpha) = Z(\alpha)$.*

Since $z \to f^{-1}zf$ is an isomorphism from $Z(\alpha) = Aut(\mathbb{I}, \boldsymbol{\alpha})$ to $f^{-1}Z(\alpha)f = Aut(\mathbb{I}, \beta)$, we get

Corollary 26 *For any two negations β and γ, $Aut(\mathbb{I}, \beta) \approx Aut(\mathbb{I}, \gamma)$.*

Of course this last corollary follows also because the two systems $(\mathbb{I}, \beta)$ and $(\mathbb{I}, \gamma)$ are isomorphic. The upshot of all this is that furnishing $\mathbb{I}$ with any negation yields a system isomorphic to that gotten by furnishing $\mathbb{I}$ with the negation $\alpha : x \to 1 - x$.

5 De Morgan systems

Let $\circ$ be a t-norm and β a negation. Then $\diamond$ defined by $x \diamond y = \beta(\beta(x) \circ \beta(y))$ defines a binary operation on $[0, 1]$ called a **t-conorm**. It has the following characterizing properties.

- $0 \diamond x = x$.
- $x \diamond y = y \diamond x$.
- $(x \diamond y) \diamond z = x \diamond (y \diamond z)$.
- $\diamond$ is increasing in each variable.

If $\diamond$ is convex and for $x \in (0, 1)$, $x \diamond x > x \diamond 0 = x$, the t-conorm $\diamond$ is **Archimedean**. All binary operations satisfying these properties come from t-norms. If $\diamond$ satisfies these properties, then $\circ$ defined by $x \circ y = \beta(\beta(x) \diamond \beta(y))$ using any negation β is a t-norm, and $x \diamond y = \beta(\beta(x) \circ \beta(y))$. If a t-norm and t-conorm are related in this way by the negation β, then $(\mathbb{I}, \circ, \beta, \diamond)$ is a **De Morgan system**, and the t-norm $\circ$ and the t-conorm $\diamond$ are said to be **dual to one another** via the negation β.

From now on, *we are going to restrict ourselves to strict t-norms*. Their duals are called **strict t-conorms**. Suppose that f is a generator of the strict t-norm $\circ$, β is a negation, and

$$\begin{aligned} x \diamond y &= \beta(\beta(x) \circ \beta(y)) \\ &= \beta f^{-1}(f\beta(x))(f\beta(y)) \\ &= (f\beta)^{-1}(f\beta(x))(f\beta(y)) \end{aligned}$$

Thus the anti-automorphism $f\beta$ and multiplication determine the strict t-conorm $\diamond$. In general, an anti-automorphism g of $\mathbb{I}$ is a **cogenerator** of a strict t-conorm $\diamond$ if $x \diamond y = g^{-1}(g(x)(g(y)))$.

It should be clear that every t-conorm has a cogenerator, and it is easy to check that g and h are cogenerators of the same t-conorm if and only if $g = rh$ for some $r \in \mathbb{R}^{+}$.

For notational reasons, we are going to adorn our operators with their generators. Thus a strict t-norm $\circ$ will be written $\circ_f$, meaning that f is a generator of $\circ$. Ordinary multiplication is $\circ_r$ for any $r \in \mathbb{R}^+$, but that will be denoted as usual by $\cdot$. A conorm with generator h will be denoted $\diamond_h$. Finally, α_f denotes the negation with generator f. We write α_1 simply as α. So a De Morgan system looks like $(\mathbb{I}, \circ_f, \alpha_g, \diamond_h)$. Being a De Morgan system implies however that $\diamond_h = \diamond_{f\alpha_g}$. Now suppose that

$$q : (\mathbb{I}, \circ_f, \alpha_g, \diamond_h) \to (\mathbb{I}, \circ_u, \alpha_v, \diamond_w)$$

is an isomorphism. Then $q \in Aut(\mathbb{I})$ and the following hold.

$$\begin{aligned} q(x \circ_f y) &= q(x) \circ_u q(y) \\ q(\alpha_g(x)) &= \alpha_v q(x) \\ q(x \diamond_h y) &= q(x) \diamond_w q(y) \end{aligned}$$

But since $x \diamond_h y = \alpha_g(\alpha_g(x) \circ_f \alpha_g(y))$ and $x \diamond_w y = \alpha_v(\alpha_v(x) \circ_u \alpha_v(y))$, if the first two equations hold, then

$$\begin{aligned} q(x \diamond_h y) &= q\left(\alpha_g(\alpha_g(x) \circ_f \alpha_g(y))\right) \\ &= \alpha_v q((\alpha_g(x) \circ_f \alpha_g(y)) \\ &= \alpha_v(q(\alpha_g(x)) \circ_u q\left(\alpha_g(y)\right)) \\ &= \alpha_v(\alpha_v(q(x)) \circ_u (\alpha_v q(y))) \\ &= q(x) \diamond_w q(y) \end{aligned}$$

Therefore to be an isomorphism, q need only be required to satisfy the first two conditions. That is, isomorphisms from $(\mathbb{I}, \circ_f, \alpha_g, \diamond_h)$ to $(\mathbb{I}, \circ_u, \alpha_v, \diamond_w)$ are the same as isomorphisms from $(\mathbb{I}, \circ_f, \alpha_g)$ to $(\mathbb{I}, \circ_u, \alpha_v)$. We will also call these systems **De Morgan systems**.

To determine the isomorphisms q from $(\mathbb{I}, \circ_f, \alpha_g)$ to $(\mathbb{I}, \circ_u, \alpha_v)$, we just note that such a q must be an isomorphism from $(\mathbb{I}, \circ_f)$ to $(\mathbb{I}, \circ_u)$ and from $(\mathbb{I}, \alpha_g)$ to $(\mathbb{I}, \alpha_v)$. Therefore, from Theorems 14 and 23, we get the following theorem.

Theorem 27 *The set of isomorphisms from* $(\mathbb{I}, \circ_f, \alpha_g)$ *to* $(\mathbb{I}, \circ_u, \alpha_v)$ *is the set*

$$\left(u^{-1}\mathbb{R}^+ f\right) \cap \left(v^{-1} Z(\alpha) g\right)$$

This intersection may be empty, of course. That is the case when the equation $u^{-1}rf = v^{-1}zg$ has no solution for $r > 0$ and $z \in Z(\alpha)$. A particular example of this is the case where $f = g = u = 1$, $v \notin Z(\alpha)$, and $v\left(\frac{1}{2}\right) = \frac{1}{2}$. Then $r = v^{-1}z$ with $r > 0$ and $z \in Z(\alpha)$. But then

$$r\left(\frac{1}{2}\right) = v^{-1}z\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^r = \frac{1}{2}$$

Thus $r = 1$, and so $v = z$. But $v \notin Z(\alpha)$. So there are De Morgan systems $(\mathbb{I}, \circ_f, \alpha_g)$ and $(\mathbb{I}, \circ_u, \alpha_v)$ which are not isomorphic. When two De Morgan systems are isomorphic, the isomorphism is unique and the situation is this.

Theorem 28 $(\mathbb{I}, \circ_f, \alpha_g) \approx (\mathbb{I}, \circ_u, \alpha_v)$ *if and only if* $(\mathbb{I}, \circ_u, \alpha_v) = (\mathbb{I}, \circ_{fh}, \alpha_{gh})$ *for some* $h \in Aut(\mathbb{I})$*, in which case* h^{-1} *is the only such isomorphism. In particular,* $(\mathbb{I}, \circ_f, \alpha_g) \approx (\mathbb{I}, \cdot, \alpha_{gf^{-1}})$.

Proof. It is easy to check that h^{-1} is an isomorphism from $(\mathbb{I}, \circ_f, \alpha_g)$ to $(\mathbb{I}, \circ_{fh}, \alpha_{gh})$. If k is such an isomorphism, then $k = u^{-1}rf = v^{-1}zg$ for some $r \in \mathbb{R}^+$ and $z \in Z(\alpha)$. Thus $u = rfk^{-1}$ and $v = zgk^{-1}$ and so $(\mathbb{I}, \circ_u, \alpha_v) = (\mathbb{I}, \circ_{fk^{-1}}, \alpha_{gk^{-1}})$. If k were distinct from h^{-1}, then kh would be a non-trivial automorphism of $(\mathbb{I}, \circ_f, \alpha_g)$. But by Proposition 3, and Theorem 27, this is impossible. ■

One implication of this theorem, taking $f = g$, is that the theory of the De Morgan system $(\mathbb{I}, \circ_f, \alpha_f)$ is the same as that of $(\mathbb{I}, \cdot, \alpha)$. More generally this holds for $(\mathbb{I}, \circ_f, \alpha_g)$ and $(\mathbb{I}, \cdot, \alpha_{gf^{-1}})$. This suggests that in applications of De Morgan systems, one may as well take the strict t-norm to be ordinary multiplication.

Corollary 29 $Aut((\mathbb{I}, \circ_f, \alpha_g)) = \{1\}$.

Corollary 30 $(\mathbb{I}, \cdot, \beta) \approx (\mathbb{I}, \cdot, \gamma)$ *if and only if* $\gamma = r^{-1}\beta r$ *for some* $r \in \mathbb{R}^+$.

Taking $\beta = \alpha$ in this last Corollary, we see that $(\mathbb{I}, \cdot, \alpha) \approx (\mathbb{I}, \cdot, \gamma)$ if and only if $\gamma = r^{-1}\alpha r$ for some $r \in \mathbb{R}^+$. So De Morgan systems isomorphic to $(\mathbb{I}, \cdot, \alpha)$ are exactly those of the form $(\mathbb{I}, \cdot, \alpha_r)$ with $r \in \mathbb{R}^+$. Negations of the form $r^{-1}\alpha r$ are **Yager negations** [13]. Thus we can state

Corollary 31 *De Morgan systems* $(\mathbb{I}, \cdot, \beta)$ *which are isomorphic to* $(\mathbb{I}, \cdot, \alpha)$ *are precisely those with* β *a Yager negation.*

We close this section with the following remark. The system $([0, 1], \wedge, \vee, ')$, where $\wedge$, $\vee$, and $'$ are max, min, and $x' = 1 - x$ forms a De Morgan algebra in the usual lattice theoretic sense. If we replace $'$, which we have been denoting by α, by any other involution β, then the systems $([0, 1], \wedge, \vee, ')$ and $([0, 1], \wedge, \vee, \beta)$ are isomorphic. Isomorphisms between these algebras are exactly the isomorphisms between $(\mathbb{I}, \alpha)$ and $(\mathbb{I}, \beta)$. There are many and these are spelled out in Theorem 23. This suggests that in applications of De Morgan systems where $\wedge$ and $\vee$ are taken for the t-norm and t-conorm, respectively, the negation may as well be $\alpha(x) = 1 - x$.

6 The non-uniqueness of negations in De Morgan systems

We have noted that a De Morgan system $(\mathbb{I}, \circ, \beta, \diamond)$ is determined by the system $(\mathbb{I}, \circ, \beta)$. Of course, it is also determined by the system $(\mathbb{I}, \beta, \diamond)$. Is it determined by $(\mathbb{I}, \circ, \diamond)$? Another way to put it is this. How unique is the negation in a De Morgan system? Suppose that β and γ are two involutions, $\circ$ is a strict t-norm, and

$$\beta(\beta(x) \circ \beta(y)) = \gamma(\gamma(x) \circ \gamma(y))$$

That is, they both give the same strict t-conorm. Then

$$\begin{aligned}\gamma\beta(\beta(x)\circ\beta(y)) &= \gamma(x)\circ\gamma(y) \\ &= \gamma\beta(\beta(x))\circ\gamma\beta(\beta(y))\end{aligned}$$

so that $\gamma\beta$ is an automorphism of $(\mathbb{I},\circ)$. Let f be a generator of $\circ$. Then automorphisms of $(\mathbb{I},\circ)$ are of the form $f^{-1}rf$ for $r>0$. Thus $\gamma\beta=f^{-1}rf$ and $\beta=\gamma f^{-1}rf$. Since β is of order 2, so is $f^{-1}\beta f$. Thus

$$\beta=\gamma f^{-1}rf=f^{-1}r^{-1}f\gamma$$

and

$$\begin{aligned}f\beta f^{-1} &= f\left(f^{-1}r^{-1}f\gamma\right)f^{-1} \\ &= r^{-1}f\gamma f^{-1}=f\gamma f^{-1}r=rf\beta f^{-1}r\end{aligned}$$

and so $rf\beta f^{-1}r=f\beta f^{-1}$. Let $\eta=f\beta f^{-1}$. Then η is an involution and $r\eta r=\eta$.

On the other hand, if η is an involution such that for some $r>0$, $r\eta r=\eta$, then it is routine to check that for any t-norm $\circ_f$, $f^{-1}\eta f$ and $f^{-1}\eta rf$ are negations which give the same t-conorm. Are there such involutions η? Yes, of course, with $r=1$. But when $r=1$, $\gamma\beta=f^{-1}rf=1$ and $\gamma=\beta$. Are there such involutions with $r\neq 1$?

For a positive real number a, let $\eta_a(x)=e^{-\frac{a}{\ln x}}$. Then η_a is an involution satisfying $r\eta_a r=\eta_a$ for *all* $r>0$. So $f^{-1}\eta_a f$ and $f^{-1}\eta_a rf$ are negations which give the same t-conorm. But it is easy to see that $\eta_a r=\eta_{\frac{a}{r}}$. So $f^{-1}\eta_a rf=f^{-1}\eta_{\frac{a}{r}}f$. We get the following theorem.

Theorem 32 *Let $\circ$ be a strict t-norm with generator f, and let a and b be positive real numbers. Then the negations $f^{-1}\eta_a f$ and $f^{-1}\eta_b f$ give the same t-conorm.*

We look a moment at the case when the t-norm is multiplication. Let $\eta_a(x)=e^{-\frac{a}{\ln s}}$ and consider the De Morgan system $(\mathbb{I},\cdot,\eta_a)$. Suppose it is isomorphic to $(\mathbb{I},\cdot,\beta)$. By the previous theorem, $\beta=\eta_a r$ for some $r\in\mathbb{R}^+$. But the isomorphism must be $r\in\mathbb{R}^+$ since it is an automorphism of $(\mathbb{I},\cdot)$. It is easy to check that $\eta_a r=\eta_{\frac{a}{r}}$ and that $r\eta_a=\eta_{ar}$. We sum up.

Corollary 33 *The De Morgan systems $(\mathbb{I},\cdot,\eta_a)$ are all isomorphic. If $(\mathbb{I},\cdot,\eta_a)\approx(\mathbb{I},\cdot,\beta)$, then $\beta=\eta_r$ for some $r\in\mathbb{R}^+$.*

7 Some comments

1. The negations η_a satisfy $r\eta_a r=\eta_a$ for *all* $r\in\mathbb{R}^+$. There are no other negations with this property. [See (3) below.] However, Professor Richard Bagby at New Mexico State University has constructed a large family of negations β such that

$r\beta r = \beta$ for a *fixed* $r \neq 1$. Thus for each one of these negations β, the negations $\gamma = f^{-1}\beta f$ and $\delta = f^{-1}\beta r f$ give the same t-conorm from the t-norm given by f. Constructing and somehow classifying all such β seems not to have been done.

2. Let $\circ$ and $\diamond$ be a strict t-norm and strict t-conorm with generator f and cogenerator g, respectively. When does there exist a negation β such that $\circ$ and $\diamond$ are dual with respect to β? This means finding an involution β such that

$$\beta(f^{-1}(f(\beta(x))f(\beta(y))) = g^{-1}(g(x)g(y))$$

This in turns means that $f\beta = rg$ for some $r > 0$. The existence of such a β then is the same as $(f^{-1}\mathbb{R}^{+}g) \cap Inv(\mathbb{I}) \neq \emptyset$, where $Inv(\mathbb{I})$ is the set of involution of $\mathbb{I}$. Consider the equation $f\beta = rg$. If f and β are given, then the dual conorm is determined, or if g and β are given, then the dual norm is determined. However, if f and g are given, then the equation must be solved for r and β, and there may be many or there may be no solutions. Of course, f and g are determined only up to left multiples of elements of $\mathbb{R}^{+}$. In the special case $f = 1$, it means for the anti-automorphism g, one must find an r such that rg is an involution, as the statement $(f^{-1}\mathbb{R}^{+}g) \cap Inv(\mathbb{I}) \neq \emptyset$ above says.

3. The involutions $\eta_a(x) = e^{-\frac{a}{\ln x}}$ all have the property that $r\eta_a$ is an involution. In fact, as we have observed, $r\eta_a = \eta_{ra}$. Also $\eta_a r = \eta_{\frac{a}{r}}$, and $\eta_a \eta_b = \frac{a}{b}$. This means that the η_a and the elements of $\mathbb{R}^{+}$ form a subgroup of $Map(\mathbb{I})$ and $\mathbb{R}^{+}$ is normal in that subgroup. The elements η_a are in the **normalizer**

$$N(\mathbb{R}^{+}) = \{g \in Map(\mathbb{I}) : g^{-1}rg \in \mathbb{R}^{+} \text{ for all } r \in \mathbb{R}^{+}\}$$

of $\mathbb{R}^{+}$. Professor Fred Richman at Florida Atlantic University has determined $N(\mathbb{R}^{+})$. There are automorphisms in $N(\mathbb{R}^{+})$ besides those in $\mathbb{R}^{+}$, but no negations η besides the η_a that satisfy $\eta r \eta = r^{-1}$. Associating elements f of $N(\mathbb{R}^{+}) \cap Aut(\mathbb{I})$ with the t-norm $\circ_f$ puts a group structure isomorphic to $(N(\mathbb{R}^{+}) \cap Aut(\mathbb{I}))/\mathbb{R}^{+}$ on those t-norms coming from $N(\mathbb{R}^{+}) \cap Aut(\mathbb{I})$. So this is a non-trivial group. These considerations will be the subject of a later article.

4. Viewing the η_a as cogenerators, they all give the same conorm since $r\eta_a = \eta_{ra}$. They give the conorm $x \diamond y = x^{\frac{\ln y}{\ln xy}}$ which doesn't look commutative but is, and $x \diamond x = \sqrt{x}$.

5. It's easy to calculate a generator for a given negation β. One generator for β is $\frac{\alpha\beta+1}{2}$. But a generator for the negation $f^{-1}\alpha f$ is f and this formula gives the more complicated $\frac{\alpha f^{-1}\alpha f+1}{2}$. Of course they differ by a composition with an element of $Z(\alpha)$. Getting a nice expression for the negation generated by an automorphism f may be impossible: it involves calculating f^{-1}.

8 Bibliography

References

[1] Abel, N., Untersuchungen der Funktionen Zweier unabhängig veränderlichen Grössen x und y, welche Eigenschaft haben, dass $f(z, f(x, y))$ eine symmetrische Funktion von x, y und z ist, *J. Reine Angew. Math.* **1** (1826), 11-15.

[2] Aczél, J., Sur les opérations définies pour nombres réels, *Bull. Soc. Math. France* **76** (1949), 59-64.

[3] Aczél, J., *Lectures on Functional Equations and Applications*, Academic Press, New York, 1966.

[4] Alsina, C., E. Trillas, and L. Valverde, On some logical connectives for fuzzy sets, 71-76, in *Fuzzy Sets for Intelligent Systems,* D. Dubois, H. Prade, and R. Yager, eds., Morgan Kaufmann, San Mateo, 1993.

[5] Dubois, D., H. Prade, and R. Yager, eds., *Fuzzy Sets for Intelligent Systems*, Morgan Kaufmann, San Mateo, 1993.

[6] Fodor, J., and M. Roubens, *Fuzzy Preference Modelling and Multicriteria Decision Support*, Kluwer, Dordrecht, 1994.

[7] Klir, G. T. and T. A. Folger, *Fuzzy Sets, Uncertainty, and Information*, Prentice Hall, Englewood Cliffs, 1988.

[8] Ling, C. H., Representation of associative functions, *Publ. Math. Debrecen* **12** (1965), 182-212.

[9] Ovchinnikov, S., and M. Roubens, On strict preference relations, *Fuzzy Sets and Systems*, **43** (1991), 319-326.

[10] Schweizer, B. and Sklar, A., Associative functions and statistical triangle inequalities, *Publ. Math. Debrecen* **8** (1961), 169-186.

[11] Schweizer, B. and Sklar, A., *Probabilistic Metric Spaces, North-Holland, Amsterdam*, 1983.

[12] Trillas, E., Sobre funciones de negación en la teoria de conjuntos difusos, *Stochastica* **III-1** (1979), 47-60.

[13] Yager, R., On a general class of fuzzy connectives, *Fuzzy Sets and Systems* **4** (1980), 235-242.