Algebraic Aspects of Fuzzy Connectives

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Abstract

The set of multiplicative generators of strict t-norms and strict t-conorms forms a group under composition—the group of automorphisms and antiautomorphisms of the ordered unit interval. In this paper we describe connections between this group and properties of systems that arise from Archimedean t-norms and t-conorms, negations, and other fuzzy connectives and collections of such.

1 Introduction

This paper is a survey of some of our work on the algebraic systems that arise in fuzzy set theory. We refer to the papers [7, 8, 9, 13, 15] for proofs and further details. We take an algebraic point of view, asking when are the various systems that arise isomorphic, and what are their symmetries (automorphisms). The bulk of the material centers around t-norms, and a typical concern is with a de Morgan system—the unit interval endowed with its natural order structure together with a t-norm and a negation.

A **fuzzy subset** A of a set S is a mapping $A : S \to [0,1]$. Operations on the set of all such fuzzy subsets of S come from operations on $[0,1]$. Standard ones are $\wedge$, $\vee$, and $'$ given by

$$(A \wedge B)(s) = \min\{A(s), B(s)\}$$
$$(A \vee B)(s) = \max\{A(s), B(s)\}$$
$$A'(s) = 1 - A(s)$$

Viewing subsets of S as mappings $S \to \{0,1\}$, these operations generalize the usual notions of intersection, union, and complement. There are many other such generalizations and a huge literature dealing with them. Our main concern will be with special classes of those connectives on the unit interval, particularly strict and nilpotent Archimedean t-norms and t-conorms, and strong negations. We give the notion of multiplicative generators a new emphasis, and investigate systematically the isomorphisms between de Morgan systems on the unit interval.

The algebra $\mathbb{I} = ([0,1], \leq)$ consisting of the unit interval with its natural order is the basic building block of fuzzy set theory.

Definition 1 *An* ***automorphism*** *of* $\mathbb{I}$ *is a one-to-one mapping* f *of* $[0,1]$ *onto* $[0,1]$ *such that* $f(a) \leq f(b)$ *if and only if* $a \leq b$*. An* ***antiautomorphism*** *of* $\mathbb{I}$ *is a one-to-one mapping* g *of* $[0,1]$ *onto* $[0,1]$ *such that* $g(a) \geq g(b)$ *if and only if* $a \leq b$*.*

Since discontinuities of monotone functions are jumps, these automorphisms and antiautomorphisms are continuous. Automorphisms are strictly increasing, and antiautomorphisms are strictly decreasing. Any continuous strictly increasing map connecting $(0,0)$ and $(1,1)$ in the plane is an automorphism of $\mathbb{I}$.

Let $Map(\mathbb{I})$ denote the set consisting of all automorphisms and all antiautomorphisms of $\mathbb{I}$, and let $Aut(\mathbb{I})$ denote the set of all automorphisms of $\mathbb{I}$. The elements of $Map(\mathbb{I})$ are functions, and may be composed: if f and g are in $Map(\mathbb{I})$, fg is the element of $Map(\mathbb{I})$ given by $(fg)(x) = f(g(x))$. With this operation, $Map(\mathbb{I})$ is a **group.** This means that

- $f(gh) = (fg)h$ (Composition is associative.)
- There is an element id in $Map(\mathbb{I})$ such that $\mathrm{id} \circ f = f \circ \mathrm{id} = f$ for all f. (The function id is the function given by $\mathrm{id}(x) = x$ for all x. It is called the **identity** of the group.)
- For each $f \in Map(\mathbb{I})$, there is an element $f^{-1} \in Map(\mathbb{I})$ such that $f \circ f^{-1} = f^{-1} \circ f = \mathrm{id}$. (The element f^{-1} is simply the inverse of f as a function on $[0,1]$.)

$Aut(\mathbb{I})$ is a **subgroup** of $Map(\mathbb{I})$: the restriction of the operation on $Map(\mathbb{I})$ to $Aut(\mathbb{I})$ makes $Aut(\mathbb{I})$ into a group. This subgroup happens to be **normal**, that is, for every element f of $Aut(\mathbb{I})$ and g of $Map(\mathbb{I})$, the **conjugate** $g^{-1}fg$ belongs to the subgroup $Aut(\mathbb{I})$. Here are some other important subgroups:

- Each positive real number r gives an automorphism of $\mathbb{I}$ by $r(x) = x^r$. Identifying r with this automorphism, the set $\mathbb{R}^+$ of positive real numbers is a subgroup of $Aut(\mathbb{I})$.
- For any subset S of $Map(\mathbb{I})$ the set $\{f \in Map(\mathbb{I}) : fs = sf \text{ for all } s \in S\}$ is the **centralizer** $Z(S)$ of S in $Map(\mathbb{I})$ and is a subgroup of $Map(\mathbb{I})$.

We will be interested only in those f which are in $Aut(\mathbb{I})$, and for any $g \in Map(\mathbb{I})$ we will write

$$Z(g) = Z(\{g\}) = \{f \in Aut(\mathbb{I}) : fg = gf\}$$

A particularly well-known antiautomorphism is $\alpha(x) = 1-x$. The group $Z(\alpha)$ consists exactly of those elements of $Aut(\mathbb{I})$ which commute with α, which is equivalent to

$$f(x) + f(1-x) = 1.$$

An easy computation establishes the following.

Theorem 2 $Z(\alpha) = \{\frac{\alpha f \alpha + f}{2} : f \in Aut(\mathbb{I})\}$.

The function $\Phi : Aut(\mathbb{I}) \to Z(\alpha) : f \to \frac{\alpha f \alpha + f}{2}$ fixes $Z(\alpha)$ elementwise, but it is not a homomorphism: $\Phi(f)\Phi(g) = \Phi(f\Phi(g)) \neq \Phi(fg)$. We will need the following proposition later on. First, notice that for any $f \in Map(\mathbb{I})$ and any subgroup G of $Map(\mathbb{I})$, the conjugate

$$f^{-1}Gf = \{f^{-1}gf : g \in G\}$$

is also a subgroup of $Map(\mathbb{I})$.

Proposition 3 *For any f and $g \in Map(\mathbb{I})$,*

$$\left(f^{-1}\mathbb{R}^{+}f\right) \cap \left(g^{-1}Z(\alpha)g\right) = \{\mathrm{id}\}$$

2 Convex Archimedean t-norms

We will put additional structure on the system $\mathbb{I}$, and first we consider t-norms. They are generalizations of intersection and are one of the fundamental objects of interest in fuzzy set theory and logic.

Definition 4 *A **t-norm** is a binary operation $\circ$ on $[0,1]$ such that for all $x, y, z \in [0,1]$, $1 \circ x = x$; $x \circ y = y \circ x$; $(x \circ y) \circ z = x \circ (y \circ z)$; $x \leq x_1$ and $y \leq y_1$ imply that $x \circ y \leq x_1 \circ y_1$.*

Thus a binary operation on $[0,1]$ is a t-norm if 1 is an identity, it is commutative, associative, and increasing in each variable. Of course, the associative property gives unambiguous meaning to $x_1 \circ x_2 \circ \cdots \circ x_n$, and in particular to $x \circ x \circ \cdots \circ x$, which we write as $x^{[n]}$, where n is the number of x's. A t-norm $\circ$ has the following additional properties.

- $0 \circ x = 0$. (This follows since $0 \circ x \leq 0 \circ 1 = 0$.)
- $x \circ y = 1$ if and only if $x = y = 1$.

Well-known examples of t-norms include minimum, multiplication, and the Łukaciewisz t-norm: $x \blacktriangle y = (x + y - 1) \vee 0$.

Definition 5 *Let $\circ$ and $\diamond$ be t-norms. The systems $(\mathbb{I}, \circ)$ and $(\mathbb{I}, \diamond)$ are **isomorphic** if there is an element $h \in Aut(\mathbb{I})$ such that $h(x \circ y) = h(x) \diamond h(y)$. We write $(\mathbb{I}, \circ) \approx (\mathbb{I}, \diamond)$. The mapping h is an **isomorphism**. The t-norms $\circ$ and $\diamond$ are **isomorphic** if $(\mathbb{I}, \circ) \approx (\mathbb{I}, \diamond)$.*

This means that the systems $([0,1], \leq, \circ)$ and $([0,1], \leq, \diamond)$ are isomorphic in the sense of universal algebra: there is a one-to-one map from $[0,1]$ onto $[0,1]$ that preserves the operations and relations involved. Isomorphism of t-norms is an equivalence relation and partitions t-norms into equivalence classes. The t-norm min is rather special. A t-norm $\circ$ is **idempotent** if $a \circ a = a$ for all $a \in [0,1]$. If $\circ$ is idempotent, then for $a \leq b$, $a = a \circ a \leq a \circ b \leq a \circ 1 = a$, so $\circ = \min$. Thus min is the *only* idempotent t-norm. It is in an equivalence class all by itself.

An isomorphism of a system with itself is called an **automorphism**. It is easy to show that the set of automorphisms of a t-norm $(\mathbb{I}, \circ)$ is a subgroup of $Aut(\mathbb{I})$. Thus, with each t-norm $\circ$, there is a group associated with it, namely its **automorphism group**

$$Aut(\mathbb{I}, \circ) = \{f \in Aut(\mathbb{I}) : f(x \circ y) = f(x) \circ f(y)\}$$

For the t-norm $a \wedge b = \min\{a, b\}$, it is clear that $Aut(\mathbb{I}, \wedge) = Aut(\mathbb{I})$. For the t-norm multiplication,

$$\begin{aligned} Aut(\mathbb{I}, \cdot) &= \{f \in Aut(\mathbb{I}) : f(xy) = f(x)f(y)\} \\ &= \left\{f \in Aut(\mathbb{I}) : f(x) = x^c, c \in \mathbb{R}^+\right\} = \mathbb{R}^+. \end{aligned}$$

This is a well-known, classical result related to one of the four basic functional equations of Cauchy (see, for example, [3]).

If H is a subgroup of a group G, and $g \in G$, then we have already noted that $g^{-1}Hg = \{g^{-1}hg : h \in H\}$ is a subgroup of G. This subgroup is said to be **conjugate to** H, or a **conjugate of** H. The map $h \to g^{-1}hg$ is an isomorphism from H to its conjugate $g^{-1}Hg$. By an **isomorphism** from a group G to a group H we mean a one-to-one onto map $f : G \to H$ such that $f(xy) = f(x)f(y)$. If $f : (\mathbb{I}, \circ) \to (\mathbb{I}, \diamond)$ is an isomorphism of two t-norms, then the map $g \to f^{-1}gf$ is an isomorphism from $Aut(\mathbb{I}, \diamond)$ to $Aut(\mathbb{I}, \circ)$, so $f^{-1}Aut(\mathbb{I}, \diamond)f = Aut(\mathbb{I}, \circ)$. This proves the following theorem.

Theorem 6 *If two t-norms are isomorphic then their automorphism groups are conjugate.*

Definition 7 *A t-norm $\circ$ is **convex** if whenever $x \circ y \leq c \leq x_1 \circ y_1$, then there is an r between x and x_1 and an s between y and y_1 such that $c = r \circ s$. A t-norm $\circ$ is **Archimedean** if for each $a, b \in (0,1)$, there is a positive integer n (where n depends on a and b) such that $a^{[n]} < b$.*

For t-norms, the condition of convexity is equivalent to continuity in the usual topology on the unit interval. We refer to the condition as convex because this formulation has the advantage of being strictly order theoretic, allowing us to remain within the algebraic context of $\mathbb{I}$ as a lattice. For convex t-norms, the condition for Archimedean simplifies, as the following well-known proposition attests.

Proposition 8 *The following are equivalent for a convex t-norm* $\circ$.

1. $\circ$ *is Archimedean.*
2. $a \circ a < a$ *for all* $a \in (0, 1)$.

The theorem below is fundamental in determining equivalences of convex Archimedean t-norms. It has usually been thought of as a theorem about representing t-norms by generators. A principle reference is [11]. The theorem in essence goes back at least to Abel [1].

Theorem 9 *If* $\circ$ *is a convex Archimedean t-norm then there is an* $a \in [0, 1)$ *and an isomorphism* $f : \mathbb{I} \to ([a, 1], \leq)$ *such that* $f(x \circ y) = \max\{f(x)f(y), a\}$ *for all* $x, y \in [0, 1]$. *Also if* $g : \mathbb{I} \to ([b, 1], \leq)$ *is another such isomorphism, then* $g(x \circ y) = \max\{g(x)g(y), b\}$ *if and only if* $f = rg$ *for some* $r > 0$.

A function f is called a (multiplicative) **generator** of the t-norm $\circ$ if $x \circ y = f^{-1}(\max\{f(x)f(y), f(0)\})$. Two functions f and g are generators of the same t-norm if and only if $f = rg$ for some $r > 0$, that is, $f(x) = (g(x))^r$ for all $x \in [0, 1]$.

A convex, Archimedean t-norm $\circ$ is **nilpotent** if for $a \neq 1$, $a^{[n]} = 0$ for some positive integer n, the n depending on a. It is clear from the theorem that $\circ$ is nilpotent if and only if $f(0) > 0$. So convex Archimedean t-norms fall naturally into two classes: nilpotent ones and those not nilpotent. Those not nilpotent are called **strict.**

2.1 Strict t-norms

The statement of the representation theorem simplifies in the strict t-norm case.

Theorem 10 *An Archimedean t-norm* $\circ$ *is strict if and only if there is an element* $f \in Aut(\mathbb{I})$ *such that* $f(x \circ y) = f(x)f(y)$. *Another element* $g \in Aut(\mathbb{I})$ *satisfies this condition if and only if* $f = rg$ *for some* $r > 0$.

So a generator of a strict t-norm $\circ$ is just an isomorphism from $Aut(\mathbb{I}, \circ)$ to $Aut(\mathbb{I}, \cdot)$. This means that a t-norm is strict if and only if it is isomorphic to multiplication.

Corollary 11 *For any strict t-norm* $\circ$, $Aut(\mathbb{I}, \circ) \approx Aut(\mathbb{I}, \cdot)$. *For any two strict t-norms* $\circ$ *and* $\diamond$, $Aut(\mathbb{I}, \circ) \approx Aut(\mathbb{I}, \diamond)$.

Call two automorphisms f and g **equivalent** if they give the same strict t-norm, and write $f \sim g$. Then $\sim$ is an equivalence relation and so induces a partition of the group $Aut(\mathbb{I})$. The members of this partition are the **right cosets** $\{\mathbb{R}^+ f : f \in Aut(\mathbb{I})\}$ of $\mathbb{R}^+$.

Corollary 12 *For an automorphism f of $\mathbb{I}$, let $\circ_f$ be given by $x \circ_f y = f^{-1}(f(x) f(y))$. Then*

$$\circ_f \to \mathbb{R}^+ f$$

is a one-to-one correspondence between the strict t-norms on $[0,1]$ and the right cosets of the subgroup $\mathbb{R}^+$ in $Aut(\mathbb{I})$.

Let $a \in (0,1)$. The set $G_a = \{f \in Aut(\mathbb{I}) : f(a) = a\}$ is a subgroup of $Aut(\mathbb{I})$ with exactly one element in each right coset $\mathbb{R}^+ f$. Thus for $f \in G_a$, the map $\circ_f \mapsto f$ is a one-to-one correspondence between the strict t-norms on $\mathbb{I}$ and the group G_a.

We know that for any two strict t-norms $\circ$ and $\diamond$, the systems $(\mathbb{I}, \circ)$ and $(\mathbb{I}, \diamond)$ are isomorphic. We spell out exactly what those isomorphisms are.

Theorem 13 *Let $\bullet_f$ and $\bullet_g$ be the strict t-norms with generators f and g, respectively. Then $h : (\mathbb{I}, \bullet_f) \to (\mathbb{I}, \bullet_g)$ is an isomorphism if and only if $h = g^{-1} r f$ for some $r > 0$. That is, the set of isomorphisms from $(\mathbb{I}, \bullet_f)$ to $(\mathbb{I}, \bullet_g)$ is the set*

$$g^{-1}\mathbb{R}^+ f = \{g^{-1} r f : r \in \mathbb{R}^+\}.$$

Corollary 14 *Let f be a generator of the strict t-norm $\circ$. Then*

$$Aut(\mathbb{I}, \circ) = f^{-1}\mathbb{R}^+ f \approx \mathbb{R}^+ \text{ and } Aut(\mathbb{I}, \cdot) = \mathbb{R}^+.$$

2.2 Nilpotent t-norms

There are two basic facts about nilpotent (convex Archimedean) t-norms: any two are isomorphic, and each has a trivial automorphism group.

Theorem 15 *Let $\circ$ and $\diamond$ be nilpotent t-norms with generators f and g respectively. Then $\circ$ and $\diamond$ are isomorphic and $g^{-1} r f$ is the unique isomorphism from $(\mathbb{I}, \circ)$ to $(\mathbb{I}, \diamond)$, where $r \in \mathbb{R}^+$ with $g(0) = (f(0))^r$.*

Corollary 16 *If $\circ$ is a nilpotent t-norm, then $Aut(\mathbb{I}, \circ) = \{\text{id}\}$, and $(\mathbb{I}, \circ) \approx (\mathbb{I}, \blacktriangle)$ where $\blacktriangle$ is the Łukaciewisz t-norm*

$$x \blacktriangle y = (x + y - 1) \vee 0$$

This leads to an alternate form of the representation theorem for nilpotent t-norms for which the generating function is a (unique) automorphism of $\mathbb{I}$.

Theorem 17 *An Archimedean t-norm* $\circ$ *is nilpotent if and only if there is an element* $f \in Aut(\mathbb{I})$ *such that* $f(x \circ y) = (f(x) + f(y) - 1) \vee 0$. *The automorphism* f *is unique, and the automorphism group of* $(\mathbb{I}, \circ)$ *is trivial.*

Call this isomorphism the **L-generator** of the nilpotent t-norm $\circ$. (If f is the L-generator of $\circ$, then αf defined by $\alpha f(x) = 1 - f(x)$ is the unique normalized additive generator of $\circ$.)

Corollary 18 *If* $\circ$ *and* $\bullet$ *are nilpotent t-norms with L-generators* f *and* u, *respectively, then* $u^{-1}f$ *is the unique isomorphism from* $(\circ, f)$ *to* $(\bullet, u)$.

The one-to-one correspondence between the set of nilpotent t-norms and the group $Aut(\mathbb{I})$ puts a group structure on the set of nilpotent t-norms, namely,

$$\blacktriangle_f * \blacktriangle_g = x \blacktriangle_{fg} y$$

where $x \blacktriangle_f y = f^{-1}((f(x) + f(y) - 1) \vee 0)$. The subgroup corresponding to the group $\mathbb{R}^+$ of nonnegative reals is the one-parameter family of nilpotent t-norms

$$x \blacktriangle_r y = ((x^r + y^r - 1) \vee 0)^{\frac{1}{r}}$$

for $r > 0$, with $\blacktriangle_r * \blacktriangle_s = \blacktriangle_{rs}$. This family of t-norms is known as the Schweizer-Sklar family.

Note that a nilpotent t-norm $\circ$ is strictly increasing whenever possible—that is, for $x, y, z \in (0, 1)$ with $y < z$, then $x \circ y < x \circ z$ if $x \circ z \neq 0$.

3 Negations

An element $f \in Map(\mathbb{I})$ has **order** n if $f^n = 1$, and n is the smallest such positive integer. If no such integer exists, the element has **infinite** order. All the elements of $Aut(\mathbb{I})$ have infinite order except 1 which has order 1. All antiautomorphisms are either of order two or of infinite order. We will only concern ourselves with what are usually called strong negations. We will call them simply negations.

Definition 19 *A* ***negation*** *(or involution) on* $\mathbb{I}$ *is an antiautomorphism of* $\mathbb{I}$ *of order two.*

Thus a negation is an order-reversing, one-to-one mapping β of $\mathbb{I}$ onto $\mathbb{I}$ such that $\beta(\beta(x)) = x$, or equivalently such that $\beta^2 = 1$. We reserve the notation α to denote the negation given by $\alpha(x) = 1 - x$.

It is a trivial fact that conjugates of negations are negations, but it is a little less trivial and a bit surprising that every negation is a conjugate of α by an automorphism.

Theorem 20 *Let β be a negation in $Map(\mathbb{I})$ and define f by*

$$f(x) = \frac{\alpha\beta(x) + x}{2}$$

Then $f \in Aut(\mathbb{I})$, and $\beta = f^{-1}\alpha f$. Moreover, $g^{-1}\alpha g = \beta$ if and only if $gf^{-1} \in Z(\alpha)$.

An automorphism f such that $\beta = f^{-1}\alpha f$ is a **generator** of β. So every negation has a generator, and we know when two elements of $Aut(\mathbb{I})$ give the same negation. This theorem seems to be due to Trillas [14] who takes as generators functions from $[0, 1]$ to $[0, \infty]$. Notice, however, that using elements of $Map(\mathbb{I})$ as generators of t-norms, t-conorms, and negations enables us to use the language of group theory and to involve only functions on $[0, 1]$. Also, we are automatically provided with natural operations between generators, being elements of the group $Map(\mathbb{I})$. To us, this is an important mathematical point.

Theorem 21 *Let β be a negation and let f be a generator of β. The map $\beta \to Z(\alpha)f$ is a one-to-one correspondence between the negations of $\mathbb{I}$ and the set of right cosets of the centralizer $Z(\alpha)$ of α.*

Two systems $(\mathbb{I}, \beta)$ and $(\mathbb{I}, \gamma)$ where β and γ are negations are **isomorphic** if there is a map $h \in Aut(\mathbb{I})$ with $h(\beta(x)) = \gamma h(x)$, that is if $h\beta = \gamma h$, or equivalently if $\beta = h^{-1}\gamma h$. Let f and g be generators of β and γ, respectively. If h is an isomorphism, then $hf^{-1}\alpha f = g^{-1}\alpha gh$ which means that

$$f^{-1}\alpha f = h^{-1}g^{-1}\alpha gh = (gh)^{-1}\alpha gh$$

Therefore, f and gh generate the same negation, and so $zf = gh$ for some $z \in Z(\alpha)$, and thus $h \in g^{-1}Z(\alpha)f$. It is easy to check that elements of $g^{-1}Z(\alpha)f$ are isomorphisms $(\mathbb{I}, \beta) \to (\mathbb{I}, \gamma)$. We have the following theorem.

Theorem 22 *Let β and γ be negations with generators f and g, respectively. Then the set of isomorphisms from $(\mathbb{I}, \beta)$ to $(\mathbb{I}, \gamma)$ is $g^{-1}Z(\alpha)f$.*

In particular, any two negations are isomorphic. For any negation β, $Aut(\mathbb{I}, \beta) = Z(\beta)$; and for any two negations β and γ, $Aut(\mathbb{I}, \beta) \approx Aut(\mathbb{I}, \gamma)$. Note that $Z(\alpha)$ plays a role for negations analogous to that of $\mathbb{R}^+$ for strict t-norms. See Corollary 12.

Corollary 23 *The set of isomorphisms from $(\mathbb{I}, \beta)$ to $(\mathbb{I}, \alpha)$ is the right coset $Z(\alpha)f$ of $Z(\alpha)$. In particular, the generator f of β is an isomorphism from $(\mathbb{I}, \beta)$ to $(\mathbb{I}, \alpha)$, and $Aut(\mathbb{I}, \beta) = f^{-1}Z(\alpha)f$.*

The upshot of all this is that furnishing $\mathbb{I}$ with any negation yields a system isomorphic to that gotten by furnishing $\mathbb{I}$ with the negation $\alpha : x \to 1 - x$.

4 de Morgan systems

Let $\circ$ be a t-norm and β a negation. Then $\diamond$ defined by $x \diamond y = \beta(\beta(x) \circ \beta(y))$ defines a binary operation on $[0,1]$ called a **t-conorm**. It is characterized by the properties that is is commutative, associative, increasing in each variable, and $0 \diamond x = x$.

If $\diamond$ is convex and for $x \in (0,1)$, $x \diamond x > x \diamond 0 = x$, the t-conorm $\diamond$ is **Archimedean**. If $\diamond$ satisfies these properties, then $\circ$ defined by $x \circ y = \beta(\beta(x) \diamond \beta(y))$, using any negation β, is a t-norm, and $x \diamond y = \beta(\beta(x) \circ \beta(y))$. If a t-norm and t-conorm are related in this way by the negation β, then $(\mathbb{I}, \circ, \beta, \diamond)$ is a **de Morgan system**, and the t-norm $\circ$ and the t-conorm $\diamond$ are said to be **dual to one another** via the negation β.

Now suppose that $(\mathbb{I}, \circ, \beta, \diamond)$ and $(\mathbb{I}, \triangle, \gamma, \bigtriangledown)$ are de Morgan systems and

$$q : (\mathbb{I}, \circ, \beta, \diamond) \to (\mathbb{I}, \triangle, \gamma, \bigtriangledown)$$

is an isomorphism. This means that $q \in Aut(\mathbb{I})$ and the following hold.

$$\begin{aligned} q(x \circ y) &= q(x) \triangle q(y) \\ q(\beta(x)) &= \gamma(q(x)) \\ q(x \diamond y) &= q(x) \bigtriangledown q(y) \end{aligned}$$

But since $x \diamond y = \beta(\beta(x) \circ \beta(y))$ and $x \bigtriangledown y = \gamma(\gamma(x) \triangle \gamma(y))$, if the first two equations hold, then so does the third. Therefore to be an isomorphism, q need only be required to satisfy the first two conditions. That is, isomorphisms from $(\mathbb{I}, \circ, \beta, \diamond)$ to $(\mathbb{I}, \triangle, \gamma, \bigtriangledown)$ are the same as isomorphisms from $(\mathbb{I}, \circ, \beta)$ to $(\mathbb{I}, \triangle, \gamma)$. We will also call these systems **de Morgan systems**.

In the following, we call f a **generator** of the t-norm $\circ$ if f is a multiplicative generator of a strict t-norm $\circ$ or the L-generator of a nilpotent t-norm $\circ$. Let f be a generator of a t-norm $\circ$, β a negation, and let $*$ denote either multiplication or the Łukaciewisz t-norm, depending on whether $\circ$ is strict or nilpotent, respectively. The corresponding t-conorm is then given by

$$\begin{aligned} x \diamond y &= \beta(\beta(x) \circ \beta(y)) \\ &= \beta\left(f^{-1}(f(\beta(x)) * f(\beta(y)))\right) \\ &= (f\beta)^{-1}(f\beta(x) * f\beta(y)) \end{aligned}$$

Thus the anti-automorphism $f\beta$ and $*$ determine the dual t-conorm $\diamond$. In general, an anti-automorphism g of $\mathbb{I}$ is a **cogenerator** of a t-conorm $\diamond$ if $x \diamond y = g^{-1}(g(x) * (g(y))$. It should be clear that every t-conorm has a cogenerator. The anti-automorphism g and h are cogenerators of the same t-conorm in the strict case if and only if $g = rh$ for some $r \in \mathbb{R}^+$; and if and only if $g = h$, in the nilpotent case.

A t-norm $\circ$ will be written $*_f$, meaning that f is a generator of $\circ$, and a conorm with cogenerator h will be denoted $*_h$. Finally, α_f denotes the negation with generator f. So a de Morgan system looks like $(\mathbb{I}, *_f, \alpha_g, *_{f\alpha_g})$, or simply $(\mathbb{I}, *_f, \alpha_g)$.

To determine the isomorphisms q from $(\mathbb{I}, *_f, \alpha_g)$ to $(\mathbb{I}, *_u, \alpha_v)$, we just note that such a q must be an isomorphism from $(\mathbb{I}, *_f)$ to $(\mathbb{I}, *_u)$ and from $(\mathbb{I}, \alpha_g)$ to $(\mathbb{I}, \alpha_v)$. Therefore, from Theorems 13 and 22 and Corollary 18 we get the following theorem.

Theorem 24 *The isomorphisms from $(\mathbb{I}, *_f, \alpha_g)$ to $(\mathbb{I}, *_u, \alpha_v)$ are*

$$\left(u^{-1}\mathbb{R}^+ f\right) \cap \left(v^{-1} Z(\alpha) g\right)$$

if $*_f$ and $*_u$ are both strict, and

$$\left(u^{-1} f\right) \cap \left(v^{-1} Z(\alpha) g\right)$$

if $*_f$ and $*_u$ are both nilpotent. This set is empty otherwise.

This intersection may be empty, even if the t-norms are both strict or both nilpotent, and examples are not difficult to find. There are pairs of strict and pairs of nilpotent de Morgan systems $(\mathbb{I}, *_f, \alpha_g)$ and $(\mathbb{I}, *_u, \alpha_v)$ that are not isomorphic. When two de Morgan systems are isomorphic, the isomorphism is unique and the situation is this.

Theorem 25 *Let $*$ denote either multiplication or the Łukaciewisz t-norm. Then $(\mathbb{I}, *_f, \alpha_g) \approx (\mathbb{I}, *_u, \alpha_v)$ if and only if $(\mathbb{I}, *_u, \alpha_v) = (\mathbb{I}, *_{fh}, \alpha_{gh})$ for some $h \in Aut(\mathbb{I})$, in which case h^{-1} is the only such isomorphism. In particular, $(\mathbb{I}, *_f, \alpha_g) \approx (\mathbb{I}, *, \alpha_{gf^{-1}})$.*

One implication of this theorem, taking $f = g$, is that the theory of the de Morgan system $(\mathbb{I}, *_f, \alpha_f)$ is the same as that of $(\mathbb{I}, *, \alpha)$. More generally this holds for $(\mathbb{I}, *_f, \alpha_g)$, $(\mathbb{I}, *_{fg^{-1}}, \alpha)$, and $(\mathbb{I}, *, \alpha_{gf^{-1}})$. This suggests that in applications of de Morgan systems, one may as well take the strict t-norm to be either ordinary multiplication or the Łukaciewisz t-norm, or take the negation to be $1 - x$ (but, of course, not both).

Corollary 26 *$Aut((\mathbb{I}, *_f, \alpha_g)) = \{\text{id}\}$. Two strict de Morgan systems $(\mathbb{I}, *, \beta)$ and $(\mathbb{I}, *, \gamma)$ are isomorphic if and only if $\gamma = r^{-1}\beta r$ for some $r \in \mathbb{R}^+$. Two nilpotent de Morgan systems $(\mathbb{I}, *, \beta)$ and $(\mathbb{I}, *, \gamma)$ are isomorphic if and only if $\gamma = \beta$.*

Among the negations $\gamma = r^{-1}\beta r$ for $r \in \mathbb{R}^+$ there is exactly one with a given fixed point $u \in (0, 1)$, so for example, there is a one-to-one correspondence between isomorphism classes of de Morgan systems with strict t-norms and de Morgan systems of the form $(\mathbb{I}, \cdot, \beta)$ where $\cdot$ is multiplication and β has fixed point $\frac{1}{2}$.

Taking $\beta = \alpha$ in this last Corollary, we see that $(\mathbb{I}, \cdot, \alpha) \approx (\mathbb{I}, \cdot, \gamma)$ if and only if $\gamma = r^{-1}\alpha r$ for some $r \in \mathbb{R}^+$. So de Morgan systems isomorphic to $(\mathbb{I}, \cdot, \alpha)$ are exactly those of the form $(\mathbb{I}, \cdot, \alpha_r)$ with $r \in \mathbb{R}^+$. Negations of the form $r^{-1}\alpha r$ are **Yager negations**. Thus de Morgan systems $(\mathbb{I}, \cdot, \beta)$ that are isomorphic to $(\mathbb{I}, \cdot, \alpha)$ are precisely those with β a Yager negation.

The system $([0,1],\wedge,\vee,')$, where $\wedge$, $\vee$, and $'$ are max, min, and $x' = 1-x$ forms a de Morgan algebra in the usual lattice theoretic sense. If we replace $'$, which we have been denoting by α, by any other involution β, then the systems $([0,1],\wedge,\vee,')$ and $([0,1],\wedge,\vee,\beta)$ are isomorphic. This suggests that in applications of de Morgan systems where $\wedge$ and $\vee$ are taken for the t-norm and t-conorm, respectively, the negation may as well be $\alpha(x) = 1-x$.

How unique is the negation in a de Morgan system? When is $(\mathbb{I},\circ,\diamond)$ a reduct of a de Morgan system? The following proposition is straightforward and applies to both strict and nilpotent de Morgan systems.

Proposition 27 *If* $(\mathbb{I},\circ,\beta,\diamond)$ *and* $(\mathbb{I},\circ,\gamma,\diamond)$ *are de Morgan systems having the same t-norm and t-conorm, then* $\gamma\beta \in Aut(\mathbb{I},\circ)$.

The following proposition shows that a de Morgan system is not determined by the t-norm and t-conorm alone. In the example, the corresponding de Morgan systems are isomorphic even though the negations are not equal, but this may not always be the case.

Proposition 28 *For a positive real number* a*, let* $\eta_a(x) = e^{\frac{a}{\ln x}}$*. The de Morgan systems* $(\mathbb{I},\cdot,\eta_a)$ *are all isomorphic. If* $(\mathbb{I},\cdot,\eta_a) \approx (\mathbb{I},\cdot,\beta)$*, then* $\beta = \eta_r$ *for some* $r \in \mathbb{R}^+$.

Let $\circ$ and $\diamond$ be a strict t-norm and strict t-conorm with generator f and cogenerator g, respectively. When does there exist a negation β such that $\circ$ and $\diamond$ are dual with respect to β? This means finding a negation β such that

$$\beta(f^{-1}(f(\beta(x))f(\beta(y))) = g^{-1}(g(x)g(y))$$

This in turns means that $f\beta = rg$ for some $r > 0$. The existence of such a β is equivalent to the existence of a negation in the set $f^{-1}\mathbb{R}^+g$. There may be many or there may be no such negations. Constructing and somehow classifying all such β seems not to have been done.

A nilpotent t-norm distributes over infinite meets and joins. These facts are needed in the proof of the following theorem which reveals an important direct connection between nilpotent t-norms and negations. See [8] for details of the proof.

Theorem 29 *A nilpotent t-norm* $\triangle$ *on* $\mathbb{I}$ *determines a negation* $\eta_\triangle$ *by the equation*

$$\eta_\triangle(x) = \bigvee \{y \in [0,1] : x \triangle y = 0\}$$

Definition 30 *If* $\triangle$ *is a binary operation on a lattice* $\mathbb{L}$ *with* 0*, an element* x^* *in* $\mathbb{L}$ *is the* $\triangle$-***pseudocomplement*** *of an element* x *if* $x \triangle y = 0$ *exactly when* $y \leq x^*$.

Theorem 29 says that for any nilpotent t-norm $\triangle$ on $\mathbb{I}$, the function $\eta_{\triangle}$ that gives the $\triangle$-pseudocomplement $\eta_{\triangle}(x)$ is a (strong) negation. It is easy to see that a t-norm $\triangle$ must be nilpotent in order for $\eta_{\triangle}$ to be a negation. Every negation is the $\triangle$-pseudocomplement of many nilpotent t-norms $\triangle$.

We have two ways to represent negations—as $\alpha_{\gamma} = \gamma^{-1}\alpha\gamma$ for automorphisms γ of $\mathbb{I}$ (or more generally, as $\eta_{\gamma} = \gamma^{-1}\eta\gamma$ for fixed η), and as the $\triangle$-pseudocomplement $\eta_{\triangle}$ of a nilpotent t-norm $\triangle$. The connection between the representations $\eta_f = f^{-1}\eta f$ ($f \in \text{Aut}(\mathbb{I})$) and $\eta_{\triangle}$ ($\triangle \in \text{Nilp}(\mathbb{I})$) for negations is the following: $\left(\eta_{\triangle}\right)_f = \eta_{\triangle_f}$.

The lattice $\mathbb{I}$ with the additional operations provided by a t-norm, t-conorm, and negation or other unary operations can satisfy properties reminiscent of axioms for Boolean algebras, and we name certain systems accordingly.

Definition 31 *Let $(\mathbb{I}, \triangle, \triangledown, *)$ be a system with t-norm $\triangle$, t-conorm $\triangledown$, and a decreasing unary operation $*$. We call $(\mathbb{I}, \triangle, \triangledown, *)$ a **Boolean system** if $(\mathbb{I}, \triangle, \triangledown, *)$ is a de Morgan system and for all elements x, y in $[0,1]$,*

$$\begin{aligned} x \triangle y &= 0 \text{ if and only if } y \leq x^* \\ x^* \triangledown x^{**} &= 1 \\ x^{**} &= x \end{aligned}$$

A Boolean system is a special case of an MV algebra (see [4]). If $\triangle$ is a strict t-norm, then the $\triangle$-pseudocomplement is always given by $0^* = 1$ and $x^* = 0$ for $x \neq 0$. Since this $\triangle$-pseudocomplement is not a negation, there are no Boolean systems with a strict t-norm. Theorem 29 established that for a nilpotent t-norm $\triangle$, the $\triangle$-pseudocomplement

$$\eta_{\triangle}(x) = \bigvee \{y \in [0,1] : x \triangle y = 0\}$$

is a negation. If η is any negation on $\mathbb{I}$ or $\mathbb{I}^{[2]}$ and $x \triangle \eta(x) = 0$, then the dual to $\triangle$ via the negation η satisfies

$$x \triangledown \eta(x) = \eta(\eta(x) \triangle \eta(\eta(x))) = \eta(\eta(x) \triangle x) = \eta(0) = 1.$$

This yields the following theorems.

Theorem 32 *If $\triangle$ is a nilpotent t-norm, then $(\mathbb{I}, \triangle, \triangledown, *)$ is a Boolean system if and only if $* = \eta_{\triangle}$ and $x \triangledown y = \eta_{\triangle}(\eta_{\triangle}(x) \triangle \eta_{\triangle}(y))$.*

In Section 2.2, we stated two basic facts about nilpotent convex Archimedean t-norms: any two are isomorphic and each has a trivial automorphism group $\text{Aut}(\mathbb{I}, \triangle)$. These two facts carry over immediately to Boolean systems, since the $\triangle$-pseudocomplement and the dual t-conorm are both naturally determined by $\triangle$.

The Boolean systems corresponding to the positive reals in the group of nilpotent t-norms with base point $\blacktriangle$ are of the form $\left(\mathbb{I}, \blacktriangle_r, \eta_{\blacktriangle_r}, \blacktriangledown_r\right)$ with

$$\begin{aligned} x \blacktriangle_r y &= ((x^r + y^r - 1) \vee 0)^{\frac{1}{r}} \\ \eta_{\blacktriangle_r} &= (1 - x^r)^{\frac{1}{r}} \\ x \blacktriangledown_r y &= ((x^r + y^r))^{\frac{1}{r}} \wedge 1 \end{aligned}$$

The one-parameter family $\blacktriangle_r$ of t-norms is well-known, and these t-norms are often paired with their duals relative to α. Members of the one-parameter family $\blacktriangledown_r$ of t-conorms are known as Yager t-conorms, the Yager t-norms being their duals relative to α.

5 Averaging operators on the unit interval

In Section 3 we used the map $\eta \mapsto \frac{\alpha\eta + \mathrm{id}}{2}$ from the set of negations to the group of automorphisms of $\mathbb{I}$, and in Section 1 we used the map $f \mapsto \frac{\alpha f \alpha + f}{2}$ from the group of automorphisms of $\mathbb{I}$ to the centralizer of α. In both cases we used an operation, averaging, that is not inherent in the lattice structure of $\mathbb{I}$. An average also provides a (continuous) scaling of the unit interval that is not provided by the lattice structure. Following our philosophy of working entirely within a well-defined algebraic system, we now consider averaging operators and add such an operator to a de Morgan system. We use the following definition which is a variant of those in the references [2, 6, 10, 12].

Definition 33 *An* ***averaging operator*** *on $\mathbb{I}$ is a binary operation $\dotplus : \mathbb{I}^2 \to \mathbb{I}$ satisfying for all $x, y \in [0,1]$,*

1. $x \dotplus y = y \dotplus x$ *($\dotplus$ is* commutative*).*
2. *$y < z$ implies $x \dotplus y < x \dotplus z$ ($\dotplus$ is* strictly increasing *in each variable).*
3. *$x \dotplus y \leq c \leq x \dotplus z$ implies there exists $w \in [y, z]$ with $x \dotplus w = c$ ($\dotplus$ is* convex*, i.e. continuous).*
4. $x \dotplus x = x$ *($\dotplus$ is* idempotent*).*
5. $(x \dotplus y) \dotplus (z \dotplus w) = (x \dotplus z) \dotplus (y \dotplus w)$ *($\dotplus$ is* bisymmetric*).*

The unit interval together with its usual order and an averaging operator $(\mathbb{I}, \dotplus)$ will be called a **mean system**.

The following properties of an averaging operator are well-known.

Proposition 34 *Let $\dotplus$ be an averaging operator. Then for each $x, y \in [0,1]$,*

1. *$x \wedge y \leq x \dotplus y \leq x \vee y$—that is, the average of x and y lies* between *x and y.*

2. *The function $A_x : \mathbb{I} \to [x \dotplus 0, x \dotplus 1] : y \longmapsto x \dotplus y$ is an isomorphism—that is, A_x is an increasing function that is both one-to-one, and onto.*

An averaging operator is a generalization of the arithmetic mean. The following representation theorem holds (see, for example, Aczél [3] page 287).

Theorem 35 *If $\dotplus$ is an averaging operator on $\mathbb{I}$, then there is a unique automorphism f of $\mathbb{I}$ such that*

$$f\left(x \dotplus y\right) = \frac{f(x) + f(y)}{2}$$

We will call f a **generator** of the operator $\dotplus$ and write $\dotplus = \dotplus_f$. From the theorem above, the generator of an averaging operator is unique.

Corollary 36 *For any averaging operator $\dotplus$, the automorphism group of $(\mathbb{I}, \dotplus)$ has only one element.*

Theorem 37 *For each averaging operator $\dotplus$ on $\mathbb{I}$, the equation*

$$x \dotplus \eta(x) = 0 \dotplus 1$$

defines a negation $\eta = \eta_{\dotplus}$ on $\mathbb{I}$ with fixed point $0 \dotplus 1$. The averaging operator is self-dual with respect to $\eta_{\dotplus}$—that is,

$$x \dotplus y = \eta_{\dotplus}\left(\eta_{\dotplus}(y) \dotplus \eta_{\dotplus}(x)\right).$$

Every homomorphism between mean systems preserves the negations associated with these means—that is, if f is the generator of $\dotplus$, then $\eta_{\dotplus} = f^{-1}\alpha f$. For this reason, **mean systems with natural negation** $\left(\mathbb{I}, \dotplus, \eta_{\dotplus}\right)$ will be often be referred to simply as **mean systems**.

There is a natural one-to-one correspondence between averaging operators and nilpotent t-norms.

Theorem 38 *The condition*

$$x \bigtriangleup y \leq z \quad \text{if and only if} \quad x \dotplus y \leq z \dotplus 1$$

determines a one-to-one correspondence between nilpotent t-norms and averaging operators, namely, given an averaging operator $\dotplus$, define $\bigtriangleup_{\dotplus}$ by

$$x \bigtriangleup_{\dotplus} y = \bigwedge \left\{ z : x \dotplus y \leq z \dotplus 1 \right\}$$

This correspondence preserves generators.

The situation with strict t-norms is somewhat more complicated. Given a de Morgan system $(\mathbb{I}, \triangle, \alpha)$, the family of t-norms $\triangle$ that satisfy the equation

$$(x \triangle y) + (x \bigtriangledown y) = x + y$$

for $x \bigtriangledown y = \alpha(\alpha(x) \triangle \alpha(x))$ are called **Frank t-norms** [5]. A de Morgan system with averaging operator $\left(\mathbb{I}, \triangle, \eta_{\dot{+}}, \dot{+}\right)$ with the compatibility property

$$(x \circ y) \dot{+} (x \diamond y) = x \dot{+} y$$

is a natural generalization of Frank t-norms. These "Frank systems" are described in detail in [8].

6 Groups of generators

Families of t-norms are typically one-parameter families, that is, are in one-to-one correspondence with some subset of the real numbers, usually the positive ones. The principal fact that we want to point out is that several of these well known families come from generators forming just a few simply expressed subgroups of $Aut(\mathbb{I})$, or cosets or conjugates of such groups. Having generators expressed as compositions of simply expressed automorphisms makes some computations easy, and suggests new families. One of these subgroups is $\mathbb{R}^+$, and others come from the normalizer $N(\mathbb{R}^+) = \{f \in Map(\mathbb{I}) : f^{-1}\mathbb{R}^+ f = \mathbb{R}^+\}$ of $\mathbb{R}^+$.

The centralizer of $\mathbb{R}^+$ is simply $\mathbb{R}^+$. The proof of this depends on the fact that the only automorphisms of the t-norm multiplication are powers—that is, are elements of $\mathbb{R}^+$ (see Corollary 14).

The proof of the following theorem is purely group theoretic. See [13] or [15] for details.

Theorem 39 *$N(\mathbb{R}^+) = \{f \in Map(\mathbb{I}) : f(x) = e^{-c(-\ln x)^k},\ c > 0,\ k \neq 0\}$.*

The set of t-norms with generators in the normalizer of $\mathbb{R}^+$ in the group $Aut(\mathbb{I})$, namely $N(\mathbb{R}^+) \cap Aut(\mathbb{I})$, carries a group structure. And the set of t-norms and t-conorms with generators in $N(\mathbb{R}^+)$ carries a group structure, as well. The group structures of $N(\mathbb{R}^+)$ and $N(\mathbb{R}^+)/\mathbb{R}^+$ are described in the following corollary.

Corollary 40 *The normalizer $N(\mathbb{R}^+)$ of $\mathbb{R}^+$ in $Map(\mathbb{I})$ is isomorphic to the group*

$$\{(c, k) : c > 0, k \neq 0\}$$

with multiplication given by

$$(c', k')(c, k) = \left(c' c^{k'}, k'k\right).$$

The subgroup $\mathbb{R}^+$ corresponds to $\{(c, 1) : c \in \mathbb{R}^+\}$ and $N(\mathbb{R}^+)/\mathbb{R}^+$ to $\{(1, k) : k \neq 0\}$. Thus the natural group structure carried by the set of t-norms and t-conorms with generators in $N(\mathbb{R}^+)$ is the multiplicative group $\mathbb{R}^*$ of the nonzero real numbers.

The group $N(\mathbb{R}^+)$ splits—that is,

$$N\left(\mathbb{R}^+\right) = \mathbb{R}^+ \times K$$

with factors $\mathbb{R}^+ = Z(\mathbb{R}^+)$ normal, and the Richman group

$$K = \left\{ f \in Map\,(\mathbb{I}) : f(x) = e^{-(-\ln x)^k},\ k \neq 0 \right\}$$

which is isomorphic to the multiplicative group $\mathbb{R}^*$ of the nonzero real numbers.

Corollary 41 *The t-norms and t-conorms with generators in $N(\mathbb{R}^+)$ are given by*

$$x \circ y = e^{-\left((-\ln x)^k + (-\ln y)^k\right)^{\frac{1}{k}}}$$

with k positive giving t-norms and k negative giving t-conorms. Ordinary multiplication is the identity element of the group of t-norms and t-conorms. The negations in $N(\mathbb{R}^+)$ are precisely the elements $e^{-c(-\ln x)^{-1}} = e^{\frac{c}{\ln x}}$, that is, the elements in $N(\mathbb{R}^+)$ with parameter $k = -1$. For $k, r > 0$, the t-norm $e^{-\left((-\ln x)^k + (-\ln y)^k\right)^{\frac{1}{k}}}$ is dual to the t-conorm $e^{-\left((-\ln x)^{-r} + (-\ln y)^{-r}\right)^{\frac{1}{-r}}}$ $-\left((-\ln x)^{-r} r + (-\ln y)^{-r}\right)^{\frac{1}{-r}}$ if and only if $r = k$, in which case they are dual with respect to precisely the negations $e^{\frac{c}{\ln x}}$ in $N(\mathbb{R}^+)$.

These t-norms were first published by Aczél and Alsina who found them as solutions to functional equations.

It turns out that many of the well known families of t-norms are very closely connected with the group N and its subgroups $\mathbb{R}^+$ and K, along with some other interesting, but not so well known, families. We are going to express the set of generators of these families of t-norms as simple combinations of these three subgroups of M, two special elements of M, and one special subset of M. These are the following:

- the subgroup $\mathbb{R}^+$ of A;
- the subgroup $N = \{e^{-c(-\ln x)^r} : c > 0,\ r \neq 0\}$ of M;
- the subgroup $K = \{e^{-(-\ln x)^r} : r \neq 0\}$ of M;
- the element $\alpha(x) = 1 - x$ of M;
- the element $f(x) = e^{-\frac{1-x}{x}}$ of A;
- The set $F = \{\frac{a^x - 1}{a - 1} : a > 0,\ a \neq 1\}$

These groups and functions are closely connected. It was mentioned earlier that $N = \mathbb{R}^+ \times K$. An easy computation shows that $f\alpha f^{-1}(x) = e^{\frac{1}{\ln x}}$ so that $f\alpha f^{-1} \in K$, and in particular, f and $f\alpha$ determine the same right cosets of K and of N. That is, $Kf = Kf\alpha$ and $Nf = Nf\alpha$. This comes into play with the Dombi t-norms. Also, in some cases, α gives the duality between pairs of t-norms and t-conorms in a family even though α is not a member of the generating group or coset. Some of the named families of t-norms (and t-conorms in some cases) are listed below, together with their sets of generators.

- **Strict t-norms with multiplicative generators**

1. The Hamacher family of t-norms: $f^{-1}\mathbb{R}^+ f$

2. The Schweizer family of t-norms: $f\mathbb{R}^+$

3. The Aczél-Alsina family of t-norms, t-conorms, and negations: K

4. The Dombi family of t-norms and t-conorms: Kf

5. The Frank family of t-norms and t-conorms: F and $F\alpha$

- **Nilpotent t-norms with L-generators**

1. The Schweizer-Sklar family of t-norms: $\mathbb{R}^+$

2. The Yager family of t-norms: $\alpha^{-1}\mathbb{R}^+\alpha$

3. The Weber family of t-norms: $\alpha F^{-1}\alpha$

Other interesting strict families are obtained with generators from $f^{-1}\mathbb{R}^+$, $\alpha\mathbb{R}^+\alpha$, $f^{-1}Kf$, and $f^{-1}Nf$, for example. And other interesting nilpotent families are obtained with generators from F and F^{-1}. Just why the generators of so many of the well known families of strict and nilpotent t-norms and are simply group theoretic combinations, for example, cosets or conjugates of the groups $\mathbb{R}^+$, K, and N is not too clear. As mentioned, the Frank family (and consequently the Weber and other nilpotent families with related generating sets) do not fit that mold. It is of interest to note that the groups K and N come from $\mathbb{R}^+$ rather directly, N being the normalizer of $\mathbb{R}^+$ and K being a complementary summand to $\mathbb{R}^+$ in N, as indicated at the beginning of this section.

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