Early Writings on Graph Theory: Topological Connections

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Introduction

The earliest origins of graph theory can be found in puzzles and game, including Euler's Königsberg Bridge Problem and Hamilton's Icosian Game. A second important branch of mathematics that grew out of these same humble beginnings was the study of position ("analysis situs"), known today as *topology*[1]. In this project, we examine some important connections between algebra, topology and graph theory that were recognized during the years from 1845 - 1930.

The origin of these connections lie in work done by physicist Gustav Robert Kirchhoff [1824 - 1887] on the flow of electricity in a network of wires. Kirchhoff showed how the current flow around a network (which may be thought of as a graph) leads to a set of linear equations, one for each circuit in the graph. Because these equations are not necessarily independent, the question of how to determine a complete set of mutually independent equations naturally arose. Following Kirchhoff's publication of his answer to this question in 1847, mathematicians slowly began to apply his mathematical techniques to problems in topology. The work done by the French mathematician Henri Poincaré [1854 - 1912] was especially important, and laid the foundations of a new subject now known as "algebraic topology."

This project is based on excerpts from a 1922 paper in which the American mathematician Oswald Veblen [1880 - 1960] shows how Poincaré formalized the ideas of Kirchhoff. An American mathematician born in Iowa, Veblen's father was also a mathematician who taught mathematics and physics at the State University of Iowa. At that time, graduate programs in mathematics were relatively young in the United States. A member of the first generation of American mathematicians to complete their advanced work in the United States rather than Europe, Oswald Veblen completed his Ph.D. at the University of Chicago in 1903. He remained in Chicago for two years before joining the mathematics faculty at Princeton. In 1930, he became the first faculty member of the newly founded Institute for Advanced Study at Princeton.[2] A talented fund-raiser and organizer, Veblen also served on the Institute's Board of Trustees in its early years.

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[1] According to Euler, the first person to discuss "analysis situs" was the mathematician and philosopher Gottfreid Leibniz [1646 - 1716]. In a 1679 letter to Christian Huygens [1629 - 1695], Leibniz wrote:

> I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful constructions of geometry. Consequently, in view of this, I consider that we need yet another kind of analysis, geometric or linear, which deals directly with position, as algebra deals with magnitude. [1, p. 30]

Although Leibniz himself did not appear to make contributions to the development of *analysis situs*, he did make important contributions to the development of another kind of analysis. Today, Leibniz is recognized alongside the mathematician and physicist Isaac Newton [1642 - 1727] as an independent co-inventor of calculus.

[2] The celebrated physicist Albert Einstein (1879 - 1955) was another early faculty member of the Institute, joining Veblen there in 1931.

During the Nazi years, Veblen was instrumental in assisting European mathematicians to find refuge in the United States. Although some American mathematicians, including George Birkhoff (1844 - 1944), voiced opposition to these efforts – fearing that talented young American mathematicians would lose academic positions to the immigrants – the Rockefeller Foundation and other philanthropic bodies provided financial support to these efforts as a means of recruiting world-class mathematicians and scientists to the United States. Veblen was also instrumental in the establishment of the American Mathematical Society's *Mathematical Reviews*, a publication aimed at providing researchers with reviews of recent mathematical papers in a timely fashion. Founded during the late 1930's when the well-known German review journal *Zentrallblatt für Mathematik und ihre Grenzgebiete* was refusing to publish reviews written by Soviet and Jewish scholars, the *Mathematical Reviews* continues to play an important role in disseminating research results and promoting communication within the mathematical community.

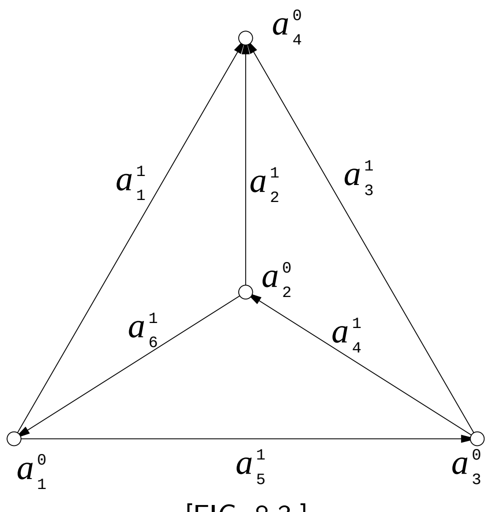
In addition to his administrative and philanthropic work, Veblen was an active researcher who made important contributions in projective and differential geometry in addition to topology, authoring influential books in all three areas. In this project, we examine extracts from his *Analysis Situs*, the first textbook to be written on combinatorial topology. Veblen first presented this work in a series of invited Colloquium Lectures of the American Mathematical Society in 1916. Although he remained interested in topology afterwards, he published little research in this area following the 1922 publication of *Analysis Situs*. The extracts we examine are taken from [1, pp. 136 - 141].

Note: This project assumes the reader is familiar with basic notions of graph theory, including the definition of *isomorphism* and *isomorphism invariant*. Parts of the project (clearly marked as such) also assume familiarity with the basic linear algebra concepts of *rank*, *kernel* and *linear independence*. As needed, the reader should refer to a standard linear algebra textbook to review these concepts.

Analysis Situs

American Mathematical Society Colloquium Lectures 1916

Symbols for Sets of Cells



[FIG. 8.3.]

14 Let us denote the 0-cells of a one-dimensional complex C_1 by $a_1^0, a_2^0, ..., a_{\alpha_0}^0$ and the 1-cells by $a_1^1, a_2^1, ..., a_{\alpha_1}^1$.

Any set of 0-cells of C_l may be denoted by a symbol $(x_l, x_2, \ldots, x_{\alpha_0})$ in which $x_i = 1$ if a_i^0 is in the set and $x_i = 0$ if a_i^0 is not in the set. Thus, for example, the pair of points a_1^0, a_4^0 in Fig. [8.3] is denoted by $(1, 0, 0, 1)$. The total number of symbols $(x_l, x_2, \ldots, x_{\alpha_0})$ is 2^{α_0}. Hence the total number of sets of 0-cells, barring the null-set, is $2^{\alpha_0} - 1$. The symbol for a null-set, $(0, 0, ..., 0)$ will be referred to as zero and denoted by 0. The marks 0 and 1 which appear in the symbols just defined, may profitably be regarded as residues, modulo 2, i.e., as symbols which may be combined algebraically according to the rules

$$\begin{aligned} 0+0=1+1=0, &\qquad 0+1=1+0=1, \\ 0\times 0=0\times 1=1\times 0=0, &\qquad 1\times 1=1 \end{aligned}$$

Under this convention the sum (mod. 2) of two symbols, or of the two sets of points which correspond to the symbols $(x_1, x_2, \ldots, x_{\alpha_0}) = X$ and $(y_1, y_2, \ldots, y_{\alpha_0}) = Y$, may be defined as $(x_1 + y_1, x_2 + y_2, \ldots, x_{\alpha_0} + y_{\alpha_0}) = X + Y$.

Geometrically, $X + Y$ is the set of all points which are in X or in Y but not in both. For example, if $X = (1, 0, 0, 1)$ and $Y = (0, 1, 0, 1)$, $X + Y = (1, 1, 0, 0)$; i.e., X represents a_1^0 and a_4^0, Y represents a_2^0 and a_4^0, and X + Y represents a_1^0 and a_2^0. Since a_4^0 appears in both X and Y, it is suppressed in forming the sum, modulo 2. This type of addition has the obvious property that if two sets contain each an even number of 0-cells, the sum (mod. 2) contains an even number of 0-cells.

15 Any set, S, of 1-cells in C_1 may be denoted by a symbol $(x_1, x_2, \ldots, x_{\alpha_1})$ in which $x_i = 1$ if a_i^1 is in the set and $x_i = 0$ if a_i^1 is not in the set. The 1-cells in the set may be thought of as labelled with 1's and those not in the set as labelled with 0's. The symbol is also regarded as representing the one-dimensional complex composed of the 1-cells of S and the 0-cells which bound them. Thus, for example, in Fig. [8.3] the boundaries of two of the faces are $(1, 0, 1, 0, 1, 0)$ and $(1, 1, 0, 0, 0, 1)$. The sum (mod. 2) of two symbols $(x_1, x_2, \ldots, x_{\alpha_1})$ is defined in the same way as for the case of symbols representing 0-cells. Correspondingly if C_1' and C_1'' are one-dimensional complexes which have a certain number (which may be zero) of 1-cells in common and have no other common points except the ends of these 1-cells, the sum $C_1' + C_1''$ (mod. 2) is defined as the one-dimensional complex obtained by suppressing all l-cells common to C' and C'' and retaining all 1-cells which appear only in C_1' or in C_1''. For example, in Fig. [8.3], the sum of the two curves represented by $(1, 0, 1, 0, 1, 0)$ and $(1, 1, 0, 0, 0, 1)$ is $(0, 1, 1, 0, 1, 1)$ which represents the curve composed of $a_2^1, a_4^1, a_5^1, a_6^1$ and their ends.

1. The following questions are based on Section 14 of Veblen's paper, in which Veblen discusses symbols for sets of 0 - cells.
   (a) In graph terminology, what is a '0-cell'? What is a '1-cell'?
   (b) How would the set of points $\{a_1^0, a_3^0, a_4^0\}$ in Fig. 8.3 be represented by Veblen?
   (c) Identify the set of points in Fig. 8.3 that are represented by the following 4-tuples:
   $$W = (0, 1, 1, 0) \qquad\qquad Z = (1, 1, 1, 1)$$

(d) Find $W+Z$ for $W=(0,1,1,0)$ and $Z=(1,1,1,1)$.
Identify the set of points in Fig. 8.3 that is represented by $W+Z$.

(e) In the last paragraph of Section 14, Veblen asserts the addition modulo 2 has a certain 'obvious property.' What property is this? How obvious is this property? Give a formal proof of the property, and interpret it in terms of sets of '0-cells'.

2. The following questions are based on Section 15 of Veblen's paper, in which Veblen discusses symbols for sets of 1 - cells.

Referring to Fig. 8.3 in Veblen's paper, let

M be the circuit defined by edges $a_1^1, a_3^1, a_4^1, a_6^1$

N be the circuit defined by edges a_2^1, a_3^1, a_4^1

(a) How would Veblen represent M and N as 6 - tuples?

(b) Find $M+N$ mod. 2, and identify the circuit represented by this sum in Fig. 8.3.

In section 16 and 17 of his article, Veblen defines two important matrices associated with a graph, denoted by him as H_0 and H_1 respectively. In section 20, he provides more detail concerning the matrix H_1 for graphs that are not connected. Project questions pertaining to each of these three sections of Veblen's paper provide additional clarification of these ideas.

16 Any one-dimensional complex falls into R_0 sub-complexes each of which is connected. Let us denote these sub-complexes by $C_1^1, C_1^2, \ldots, C_1^{R_0}$ and let the notation be assigned in such a way that a_i^0 $(i=1,2,...,m_1)$ are the 0-cells of C_1^1, a_i^0 $(i=m_1+1, m_1+2, ..., m_2)$ those of C_1^2, and so on.

With this choice of notation, the sets of vertices of $C_1^1, C_1^2, \ldots, C_1^{R_0}$, respectively, are represented by the symbols $(x_1, x_2, \ldots, x_{\alpha_0})$ which constitute the rows of the following matrix.

$$H_0 = \left\| \begin{matrix} \overbrace{1\,1\ldots1}^{m_1} & \overbrace{0\,0\ldots0}^{m_2-m_1} & \overbrace{0\,0\ldots0}^{\alpha_0-m_{R_0-1}} \\ 0\,0\ldots0 & 1\,1\ldots1 & 0\,0\ldots0 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ 0\,0\ldots0 & 0\,0\ldots0 & 1\,1\ldots1 \end{matrix} \right\| = \|\eta_{ij}^0\|$$

For most purposes it is sufficient to limit attention to connected complexes. In such cases $R_0=1$, and H_0 consists of one row all of whose elements are 1.

3. The following questions are based on Section 16 of Veblen's paper, in which Veblen defines the matrix H_0.

(a) In the first paragraph of this section, Veblen assumes that we may label the '0-cells of a one-dimensional complex' in a particular way. What graph isomorphism invariant allows him to make this assumption?

(b) Find the matrix H_0 for graphs G_1 and G_2 in the appendix. Using Veblen's notation to label the vertices and edges of each graph so that the correspondence to the associated matrix H_0 is clear.

Continuing with Section 17, we consider Veblen's initial discussion of the matrix H_1:

17 By the definition ... a 0-cell is incident with a 1-cell if it is one of the ends of the 1-cell, and under the same conditions the 1-cell is incident with the 0-cell. The incidence relations between the 0-cells and the 1-cells may be represented in a table or matrix of α_0 rows and α_1 columns as follows: The 0-cells of C_1 having been denoted by a_i^0, $(i = 1, 2, ..., \alpha_0)$ and the 1-cells by a_j^1, $(j = 1, 2, ..., \alpha_1)$, let the element of the ith row and the jth column of the matrix be 1 if a_i^0 is incident with a_j^1 and let it be 0 if a_i^0 is not incident with a_j^1. For example, the table for the linear graph of Fig. [8.3] formed by the vertices and edges of a tetrahedron is as follows:

	α_1^1	α_2^1	α_3^1	α_4^1	α_5^1	α_6^1
α_1^0	1	0	0	0	1	1
α_2^0	0	1	0	1	0	1
α_3^0	0	0	1	1	1	0
α_4^0	1	1	1	0	0	0

In the case of the complex used ... to define a simple closed curve the incidence matrix is

$$\left\| \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \right\|$$

We shall denote the element of the ith row and jth column of the matrix of incidence relations between the 0-cells and 1-cells by η_{ij}^1 and the matrix itself by

$$\|\eta_{ij}^1\| = H_1$$

The ith row of H_1 is the symbol for the set of all 1-cells incident with a_i^0 and the jth column is the symbol for the set of all 0-cells incident with a_j^1.

The condition which we have imposed on the graph, that both ends of every 1-cell shall be among the α_0 0-cells, implies that every column of the matrix contains exactly two I's. Conversely, any matrix whose elements are 0's and 1's and which is such that each column contains exactly two 1's can be regarded as the incidence matrix of a linear graph. For to obtain such a graph it is only necessary to take α_0 points in a 3-space, denote them arbitrarily by $a_l^0, a_2^0, ..., a_{\alpha_0}^0$, and join the pairs which correspond to 1's in the same column successively by arcs not meeting the arcs previously constructed.

4. The following questions are based on Section 17 of Veblen's paper, in which Veblen defines the incidence matrix H_1.

(a) Find the incidence matrix H_1 for graph G_3 in the appendix.

(b) Sketch and label a graph with incidence matrix $H_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$.

(c) Give two reasons why no graph can have incidence matrix $H_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$.

(d) Explain why isomorphic graphs do not necessarily have the same incidence matrix. Is there some way in which we could use incidence matrices to determine if two graphs are isomorphic? Explain.

In Section 20 of his article, Veblen provides a more detailed analysis of the form and properties of H_1 for graphs that are not connected.

20 Denoting the connected sub-complexes of C_1 by $C_1^1, C_1^2, \ldots, C_1^{R_0}$ as in 16 let the notation be so assigned that $a_1^1, a_2^1, \ldots, a_{m_1}^1$ are the 1-cells in C_1, $a_{m_1+1}^1$, $a_{m_1+2}^1$, $\ldots, a_{m_2}^1$ the 1-cells in C_2; and so on. The matrix H_1 then must take the form

I	0	0	0	
0	II	0	0	
0	0	III	0	

where all the non-zero elements are to be found in the matrices I, II, III, etc., and I is the matrix of C_1, II of C_2, etc. This is evident because no element of one of the complexes C_1^i is incident with any element of any of the others.

There are two non-zero elements in each column of H_1. Hence if we add the rows corresponding to any of the blocks I, II, etc. the sum is zero (mod. 2) in every column. Hence the rows of H_1 are connected by R_0 linear relations.

Any linear combination (mod. 2) of the rows of H_1 corresponds to adding a certain number of them together. If this gave zeros in all the columns it would mean that there were two or no 1's in each column of the matrix formed by the given rows, and this would mean that any 1-cell incident with one of the 0-cells corresponding to these rows would also be incident with another such 0-cell. These 0-cells and the 1-cells incident with them would therefore form a sub-complex of C_1 which was not connected with any of the remaining 0-cells and 1-cells of C_1. Hence it would consist of one or more of the complexes C_l^i $(i = 1, 2, ..., R_0)$ and the linear relations with which we started would be dependent on the R_0 relations already found. Hence there are exactly R_0 linearly independent linear relations among the rows of H_1, so that if ρ_1 is the rank of H_1,

$$\rho_1 = \alpha_0 - R_0.$$

5. The following questions are based on `Section 20` of Veblen's paper, in which he discusses the matrix H_1 for graphs that are not connected.

   (a) In the first paragraph of this section, Veblen asserts that the matrix H_1 must take the form of a block diagonal matrix. Illustrate this by finding the matrix H_1 for graphs G_1 and G_2 in the appendix. Then explain in general why this must be true.

   (b) In the second paragraph of this section, Veblen asserts that the rows of matrix H_1 are related by R_0 linear relations. What does the value R_0 represent? Determine the R_0 linear relations for the incidence matrices H_1 of graphs G_1 and G_2 from part (a) above. Suggestions: Denote the i^{th} row of H_1 by z_i. You may also wish to review Section 14 of Veblen's paper in which he explains the notation '0' as it applies to the representation of sets of 0 - cells.

   (c) Verify the relationship $\rho_1 = \alpha_0 - R_0$ for the incidence matrices of graphs G_1 and G_2.

   Note: You will need to know how to find the rank of a matrix to complete this question; as required, review this concept in a linear algebra textbook. Since all sums are modulo 2, you will also need to reduce the matrices to determine their rank by hand, rather than using the matrix utility on your calculator or some other computing device.

Returning to Veblen's paper, we find the introduction of the concept of a 'one-dimensional circuit.'

One-dimensional Circuits

22 A connected linear graph each vertex of which is an end of two and only two 1-cells it called a one-dimensional circuit or a 1-circuit. Any closed curve is decomposed by any finite set of points on it into a 1-circuit. Conversely, it is easy to see that the set of all points on a 1-circuit is a simple closed curve. It is obvious, further, that any linear graph such that each vertex is an end of two and only two 1-cells is either a 1-circuit or a set of 1-circuits no two of which have a point in common.

Consider a linear graph C_1 such that each vertex is an end of an even number of edges. Let us denote by $2n_i$ the number of edges incident with each vertex at a_i^0. The edges incident with each vertex a_i^0 may be grouped arbitrarily in n_i pairs no two of which have an edge in common; let these pairs of edges be called the pairs associated with the vertex a_i^0. Let C_1' be a graph coincident with C_1 in such a way that (1) there is one and only one point of C_1' on each point of C_1 which is not a vertex and (2) there are n_i vertices of C_1' on each vertex a_i^0 of C_1 each of these vertices of C_1' being incident only with the two edges of C_1' which coincide with a pair associated with at a_i^0.

The linear graph C_1' has just two edges incident with each of its vertices and therefore consists of a number of 1-circuits. Each of these 1-circuits is coincident with a 1-circuit of C_1, and no two of the 1-circuits of C_1 thus determined have a 1-cell in common. Hence C_1 consists of a number of 1-circuits which have only a finite number of 0-cells in common.

It is obvious that a linear graph composed of a number of closed curves having only a finite number of points in common has an even number of 1-cells incident with each vertex. Hence a *necessary and sufficient condition that C_1 consist of a number of*

1-circuits having only 0-cells in common is that each 0-cell of C_1 be incident with an even number of 1-cells. A set of 1-circuits having only 0-cells in common will be referred to briefly as a set of 1-circuits.

6. The following questions are based on Section 22, in which Veblen discusses one-dimensional circuits.

   (a) First paragraph: Note that Veblen is now only interested in connected graphs. According to his definition of '1-circuit', can a 1-circuit repeat edges? vertices?

   (b) Second paragraph: Veblen outlines a method for constructing a new graph C_1' from a given linear graph C_1.

      i. What conditions on the graph C_1 are required for this construction?

      ii. The resulting graph C_1' will depend on how we pair up the edges at each vertex. Illustrate this fact using graph G_4 from the appendix. That is, apply Veblen's construction method *twice* to graph G_4, using different pairings of edges each time.

   (c) Third paragraph: Veblen claims that each 1-circuit of the graph C_1' constructed by this method will be coincident with a 1-circuit of the original graph C_1.

   Use graph G_5 from the appendix to show that this is not the case for any arbitrary pairing of edges in the original graph. That is, find a pairing of the four edges at vertex D of graph G_5 which gives us a 1-circuit of the graph C_1' that is not coincident with a 1-circuit of the graph C_1'.

   (d) In the final paragraph of Section 22, Veblen states the conclusion of this section in the form of a 'necessary and sufficient' statement. To what (familiar) theorem from graph theory is this conclusion related? Explain, and comment on Veblen's proof (especially in light of question 5c above).

In Section 24 of *Analysis Situs*, Veblen introduced an algebraic representation of one-dimensional circuits.

24 Let us now inquire under what circumstances a symbol $(x_1, x_2, \ldots, x_{\alpha_1})$ for a one-dimensional complex contained in C_1 will represent a 1-circuit or a system of 1-circuits. Consider the sum

$$\eta_{i1}^1 x_1 + \eta_{i2}^1 x_2 + \ldots + \eta_{i\alpha_1}^1 x_{\alpha_1}$$

where the coefficients η_{ij}^1 are the elements of the ith row of H_1. Each term $\eta_{ij}^1 x_j$ of this sum is 0 if a_j^1 is not in the set of 1-cells represented by $(x_1, x_2, \ldots, x_{\alpha_1})$ because in this case $x_j = 0$; it is also zero if a_j^1 is not incident with a_i^0 because $\eta_{ij}^1 = 0$ in case. The term $\eta_{ij}^1 x_j = 1$ if a_j^1 is incident with a_i^0 and in the set represented by $(x_1, x_2, \ldots, x_{\alpha_1})$ because in this case $\eta_{ij}^1 = 1$ and $x_j = 1$. Hence there are as many non-zero terms in the sum as there are 1-cells represented by $(x_1, x_2, \ldots, x_{\alpha_1})$ which are incident with a_i^0. Hence by §22 the required condition is that the number of non-zero terms in the sum must be even. In other words if the x's and η_{ij}^1's are reduced modulo 2 as explained in §14 we must have

$$(H_1) \qquad \sum_{i=1}^{\alpha_1} \eta_{ij}^1 x_j = 0 \qquad (i = 1, 2, \ldots, \alpha_0)$$

if and only if $(x_1, x_2, \ldots, x_{\alpha_1})$ represents a 1-circuit or set of 1-circuits. The matrix of this set of equations (or congruences, mod. 2) is H_1.

7. Note that, in the algebraic representation of one-dimensional circuits discussed in Section 24, the number of non-zero terms in the sum $\eta_{i1}^1 x_1 + \eta_{i2}^1 x_2 + \ldots \eta_{i\alpha_1}^1 x_{\alpha_1}$ corresponds to the degree of the vertex a_i^0.

   (a) Veblen's conclusion in the penultimate sentence of Section 24 could be re-stated in terms of solutions to the matrix-vector equation $H_1 \vec{v} = \vec{0}$, where H_1 is the incidence matrix of the graph. Complete the following example to illustrate this conclusion:
   Let H_1 be the incidence matrix for graph G_3 from the appendix. (See part a of question 4 above.) Let $\vec{X} = (1, 0, 1, 1, 0)$ and $\vec{Y} = (1, 0, 0, 1, 1)$. Determine the matrix-vector products (modulo 2) $H_1\vec{X}$ and $H_1\vec{Y}$. Use these results to determine whether (i) $\vec{X}$ represents a set of 1-circuits in the graph G_3; and (ii) $\vec{Y}$ represents a set of 1-circuits in the graph G_3. Explain.

   (b) What advantage might there be in representing graphs and circuits in terms of matrices and linear equations?

In the final excerpt from Veblen's paper below, a connection is made between the matrix H_1 and the problem of determining whether there exists a complete set of 1-circuits that will generate all possible 1-circuits for a given graph.

25 If the rank of the matrix H_1 of the equations (H_1) be ρ_1 the theory of linear homogeneous equations (congruences, mod. 2) tells us that there is a set of $\alpha_1 - \rho_1$ linearly independent solutions of (H_1) upon which all other solutions are linearly dependent. This means geometrically that *there exists a set of $\alpha_1 - \rho_1$ 1-circuits or systems of 1-circuits from which all others can be obtained by repeated applications of the operation of adding (mod. 2) described in §14.* We shall call this a complete set of 1-circuits or systems of 1-circuits.

8. The following questions are based primarily on Section 25, in which Veblen discusses how to use the matrix H_1 to determine if there is a complete set of 1-circuits that will generate all possible 1-circuits for a given graph.

   Note: You will need to know about null space, basis sets and rank to complete this question; as required, review this concept in a linear algebra textbook.

(a) Explain Veblen's conclusion in terms of null space of the matrix H_1. You may find it helpful to review section 24 of Veblen's paper, and project question 7 above.

(b) Consider the incidence matrix H_1 for graph G_6 in the appendix.

- Find a basis for the null space of that matrix, again using modulo 2 sums.
- Use your null-space basis to identify a complete system of 1-circuits for this graph.
- Write the circuit $C = (0, 0, 1, 1, 1, 0, 1)$ as the sum of circuits in your complete system of 1-circuits for this graph.

Notes to the Instructor

This project was developed for use in a beginning-level discrete mathematics course; it could also be used in a more advanced level discrete mathematics course. Familiarity with basic notions of graph theory, including the definition of *isomorphism* and *isomorphism invariant*, is assumed. Parts of the project (clearly marked as such) also assume familiarity with the basic linear algebra concepts of *rank*, *kernel* and *linear independence*. The instructor may either omit these questions or refer students to a standard linear algebra textbook as needed for review. In view of the more advanced nature of this project, small group work is recommended for its use within a beginning level course. Although the project could be assigned independently, assigning it following completion of one or both of the projects "Early Writings on Graph Theory: Hamiltonian Circuits and The Icosian Game" and "Early Writings on Graph Theory: Topological Connections" is also recommended for beginning level courses; both projects appear in the current volume. Use of color pencils or markers will be helpful in answering Question 6.

References

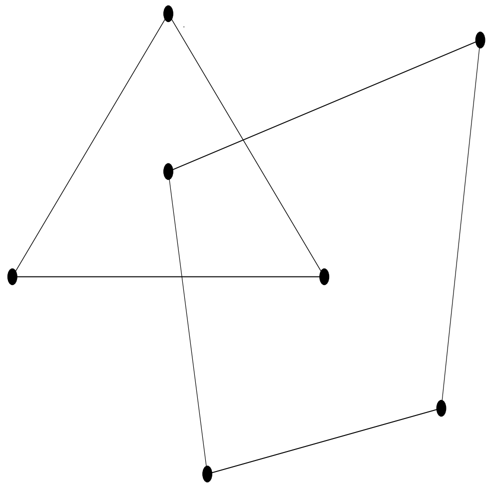
[1] Biggs, N., Lloyd, E., Wilson, R., *Graph Theory: 1736–1936*, Clarendon Press, Oxford, 1976.

[2] James, I., *Remarkable Mathematicians: From Euler to von Neumann*, Cambridge University Press, Cambridge, 2002.

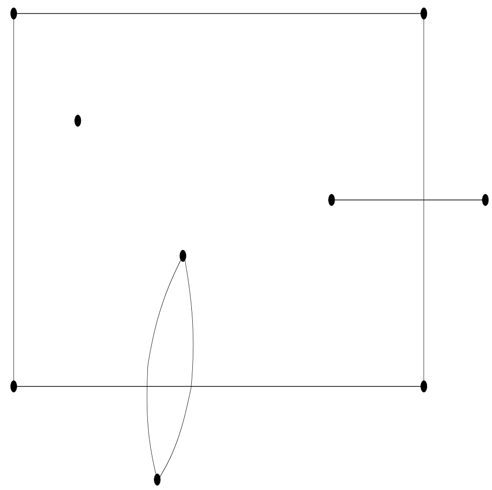
[3] Katz, V., *A History of Mathematics: An Introduction*, Second Edition, Addison-Wesley, New York, 1998.

[4] Veblen, O., "An Application of Modular Equations in Analysis Situs," *Ann. of Math.*, **14** (1912-13), 42–46.

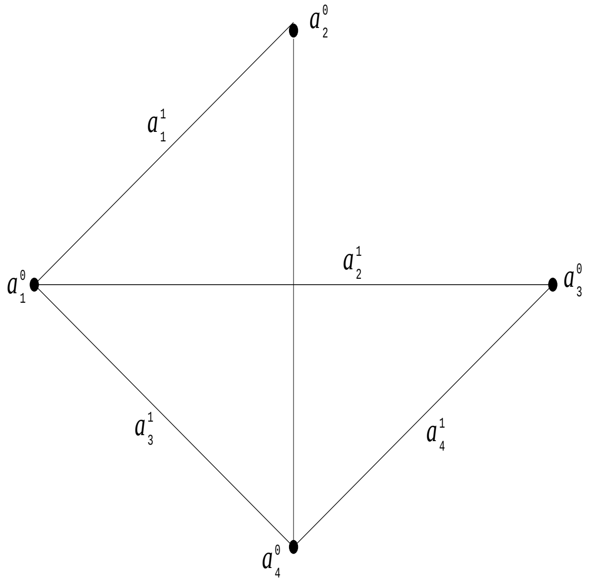
APPENDIX: Graphs for Veblen Graph Theory Project



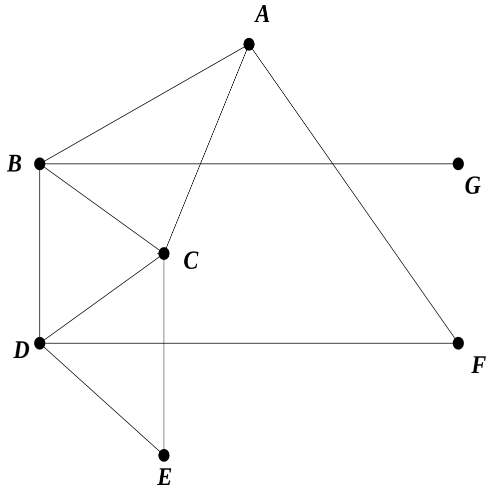
Graph G_1



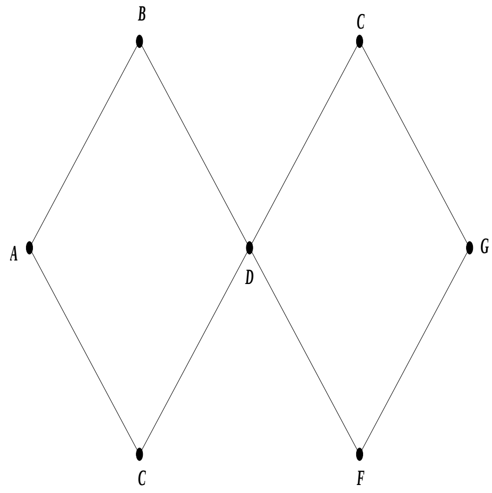
Graph G_2



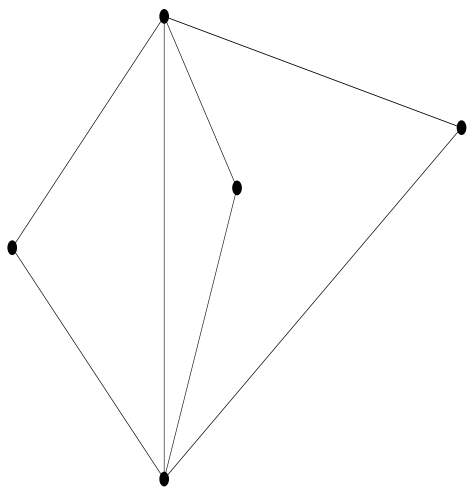
Graph G_3



Graph G_4



Graph G_5



Graph G_6