

Gabriel Lamé's Counting of Triangulations

An Historical Project

1 Introduction

In a letter to Joseph Liouville (1809–1882) in 1838, Gabriel Lamé (1795–1870) offered an elegant and efficient counting argument for the number of triangulations of a convex n -sided polygon [4]. The French mathematician, engineer and physicist Lamé was educated at the prestigious Ecole Polytechnique and later at the Ecole des Mines [3, p. 601–602]. From 1832 to 1844 he served as the chair of physics at the Ecole Polytechnique, and in 1843 joined the Paris Academy of Sciences in the geometry section. He contributed to the fields of differential geometry, number theory, thermodynamics and applied mathematics. Among his publications are textbooks in physics and papers on heat transfer, where he introduced the rather useful technique of curvilinear coordinates. In 1851 he was appointed Professor of Mathematical Physics and Probability at the University of Paris, and resigned eleven years later after becoming deaf. Gauss considered Lamé the foremost French mathematician of his day. [3, p. 601–602].

The triangulation problem, studied earlier by the prolific Leonhard Euler (1707–1783), can be stated as follows. Given a convex n -sided polygon, divide it into triangles by drawing *non-intersecting* diagonals connecting some of the vertices of the polygon. Euler calculated the number, P_n , of distinct triangulations of a convex n -gon for the first few values of n , and conjectured a formula for P_n based on an empirical study of the ratios P_{n+1}/P_n [1, p. 339–350] [2]. Lamé was one of the first to provide the details for a combinatorial proof of Euler's conjectured result for P_{n+1}/P_n , a proof which the reader will study in its original (translated) version in this project.

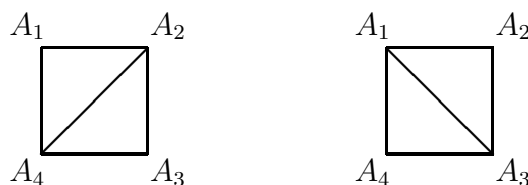
Although Euler does not state his motivation for studying the triangulation problem, it may have roots in surveying, where a given region to be surveyed is divided into triangles, with the three vertices of the triangle

serving as reference points. A modern use of triangulation is the Global Positioning System, where readings from three satellites are used to determine the position of a point on earth. For this project, however, we will consider polygons \mathcal{P} in the plane with the property that if A and B are points in \mathcal{P} , then the line segment connecting A and B is also contained in \mathcal{P} . This latter property is expressed by stating that \mathcal{P} is convex. As a further simplification, we will often use regular polygons, which have all sides congruent and all angles congruent.

Lamé, and not Euler, uses the subscripted notation P_n to denote the number of triangulations of a convex polygon with n sides.

1.1. Explain why $P_3 = 1$.

To determine P_4 , consider the following triangulations of square $A_1A_2A_3A_4$.



1.2. Can you find any other ways to triangulate square $A_1A_2A_3A_4$ with non-intersecting diagonals? What is the value of P_4 ?

1.3. Let $A_1A_2A_3A_4A_5$ be a regular pentagon and compute P_5 , the number of distinct triangulations of the pentagon. Can you identify how the number of triangulations of a convex quadrilateral (or square) enters into the calculation of the triangulations of the pentagon? Consider triangulations of the pentagon which contain the triangle $A_1A_2A_3$ first, then triangle $A_1A_2A_4$, then triangle $A_1A_2A_5$. Write an equation for P_5 in terms of P_4 and P_3 .

1.4. Let $A_1A_2 \dots A_6$ be a regular hexagon and devise a strategy for computing P_6 by using the previous results for a pentagon, quadrilateral, and a triangle. Be sure to explain your approach. What is the value of P_6 ?

1.5. Find a recursion relation for P_{n+1} in terms of the previous P_k 's. Be sure to justify your answer. For what values of k must P_k be known in order to compute P_{n+1} ? Compare your result with §I from Lamé's letter [4] in the next part.

2 Lamé's Letter to Liouville

Extrait d'une lettre de M. Lamé à M. Liouville sur cette question:
*Un polygone convexe étant donné, de combien de manières peut-on le partager
en triangles au moyen de diagonales?*¹

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**Excerpt from a letter of Monsieur Lamé to Monsieur Liouville on
the question:** *Given a convex polygon, in how many ways can one partition it
into triangles by means of diagonals?*

"The formula that you communicated to me yesterday is easily deduced from
the comparison of two methods leading to the same goal.

"Indeed, with the help of two different methods, one can evaluate the number
of decompositions of a polygon into triangles: by consideration of the sides, or
of the vertices.

I.

"Let $ABCDEF\dots$ be a convex polygon of $n + 1$ sides, and denote by
the symbol P_k the total number of decompositions of a polygon of k sides into
triangles. An arbitrary side AB of $ABCDEF\dots$ serves as the base of a triangle,
in each of the P_{n+1} decompositions of the polygon, and the triangle will have
its vertex at C , or D , or $F\dots$; to the triangle CBA there will correspond P_n
different decompositions; to DBA another group of decompositions, represented
by the product P_3P_{n-1} ; to EBA the group P_4P_{n-2} ; to FBA , P_5P_{n-3} ; and so
forth, until the triangle ZAB , which will belong to a final group P_n . Now, all

¹See a Memoir of Segner (*Novi Commentarii Acad. Petrop.*, vol. VII, p. 203). The
author found equation (1) of M. Lamé; but formula (3) presents a much simpler solution.
Formula (3) is no doubt due to Euler. It is pointed out without proof on page 14 of the
volume cited above. The equivalence of equations (1) and (3) is not easy to establish. M.
Terquem proposed this problem to me, achieving it with the help of some properties of
factorials. I then communicated it to various geometers: none of them solved it; M. Lamé
has been very successful: I am unaware of whether others before him have obtained such
an elegant solution. J. LIOUVILLE

these groups are completely distinct: their sum therefore gives P_{n+1} . Thus one has

$$(1) \quad P_{n+1} = P_n + P_3P_{n-1} + P_4P_{n-2} + P_5P_{n-3} + \cdots + P_{n-3}P_5 + P_{n-2}P_4 + P_{n-1}P_3 + P_n.$$

2.1. Explain why the triangulations belonging to the groups

$$P_n, P_3P_{n-1}, P_4P_{n-2}, \dots, P_{n-1}P_3, P_n$$

are distinct.

2.2. Does every triangulation of a convex polygon with $n + 1$ sides occur in one of the groups represented by

$$P_n, P_3P_{n-1}, P_4P_{n-2}, \dots, P_{n-1}P_3, P_n ?$$

Why or why not?

2.3. Use Lamé's recursion relation of §I to compute P_{10} . What difficulties do you encounter in this computation?

In §I of his letter, Lamé uses triangle $A_1A_2A_k$ to divide the $(n + 1)$ -sided polygon $A_1A_2A_3 \dots A_{n+1}$ into two subpolygons. The triangulations for the subpolygons then figure into the recursion relation for P_{n+1} . In §II Lamé changes his point of view, and uses instead the diagonal A_1A_k to divide the n -gon $A_1A_2A_3 \dots A_n$ into two subpolygons. Let's examine the consequences of using a diagonal instead of a triangle to divide the polygon.

2.4. Consider a regular pentagon $A_1A_2A_3A_4A_5$. Using diagonal A_1A_3 to split the pentagon into two figures, how many resulting triangulations of the original pentagon are there? Let T_1 denote the set of these triangulations. Using diagonal A_1A_4 , how many triangulations of the pentagon are there? Let T_2 denote the set of these triangulations. Compute the sum of the cardinality (the number of elements) of T_1 and T_2 , and let S_5 denote this value. How does S_5 compare to P_5 ? Are all elements of T_1 and T_2 distinct? Is every possible triangulation of the original pentagon an element of $T_1 \cup T_2$? Justify your answers.

2.5. Consider a regular hexagon $A_1A_2A_3 \dots A_6$. Let T_1 be the set of all triangulations of the hexagon that are formed using the diagonal A_1A_3 . Let

T_2 be the set of triangulations of the hexagon using A_1A_4 , and T_3 the set of triangulations using A_1A_5 . Compute

$$S_6 = |T_1| + |T_2| + |T_3|,$$

where $|T_i|$ denotes the cardinality of T_i . Do you recognize this sum? How does S_6 compare to P_6 ? Are all elements of T_1 and T_2 distinct? Is every triangulation of the original hexagon an element of $T_1 \cup T_2 \cup T_3$? Justify your answers.

2.6. Consider now a regular n -gon $A_1A_2A_3 \dots A_{n-1}A_n$. Let T_i be the set of all triangulations of the n -gon which are formed using the diagonal A_1A_{i+2} for $i = 1, 2, 3, \dots, n-3$. Write an algebraic expression for

$$S_n = |T_1| + |T_2| + |T_3| + \dots + |T_{n-3}|$$

in terms of the P_k 's. How does S_n compare to P_n ? Are the sets T_1 and T_2 disjoint? Is every triangulation of the original n -gon an element of

$$T_1 \cup T_2 \cup T_3 \cup \dots \cup T_{n-3}?$$

Justify your answer.

Let's now read from §II of Lamé's letter.

II.

"Let $abcde \dots$ be a polygon of n sides. To each of the $n-3$ diagonals, which end at one of the vertices a , there will correspond a group of decompositions, for which this diagonal will serve as the side of two adjacent triangles: to the first diagonal ac corresponds the group P_3P_{n-1} ; to the second ad corresponds P_4P_{n-2} ; to the third ae , P_5P_{n-3} , and so forth until the last ax , which will occur in the group P_3P_{n-1} . These groups are not totally different, because it is easy to see that some of the partial decompositions, belonging to one of them, is also found in the preceding ones. Moreover they do not include the partial decompositions of P_n in which none of the diagonals ending in a occurs.

2.7. In the above statement, what groups is Lamé referring to by "[T]hese groups are not totally different"? What notation have we used for "diagonals ending in a "?

Lamé's use of diagonals leads to an enumeration of triangulations which is neither one-to-one nor inclusive of all triangulations. His genius, however, was to slightly alter this strategy to first include all triangulations, and then to count how many times a generic triangulation occurs. Combined with the results of §I, this results in a streamlined computation for P_n .

2.8. Returning to pentagon $A_1A_2A_3A_4A_5$, recall that S_5 counts with certain repetitions the number of triangulations arising from diagonals A_1A_3 and A_1A_4 . How many triangulations, counting possible repetitions, would occur if diagonals A_2A_4 and A_2A_5 are used? Denote this number by $S_5^{(2)}$. How many triangulations, counting possible repetitions, would occur if diagonals A_3A_5 and A_3A_1 are used? Denote this number by $S_5^{(3)}$. Similarly, let $S_5^{(4)}$ denote the number of triangulations with repetition formed by diagonals A_4A_1 and A_4A_2 . Compute $S_5^{(4)}$. What diagonals would be used to define $S_5^{(5)}$? Compute $S_5^{(5)}$ and $\sum_{i=1}^5 S_5^{(i)}$, where $S_5^{(1)} = S_5$.

2.9. Let \mathcal{T} be an arbitrary triangulation of the pentagon. Must \mathcal{T} be included in the count

$$S_5^{(1)} + S_5^{(2)} + S_5^{(3)} + S_5^{(4)} + S_5^{(5)} ?$$

Justify your answer. How many times does \mathcal{T} occur in the sum $\sum_{i=1}^5 S_5^{(i)}$? Why? Use this number of duplications to find integers K and L with

$$K \cdot S_5 = L \cdot P_5$$

and justify your answer.

2.10. For the hexagon, consider numbers $S_6^{(1)}, S_6^{(2)}, \dots, S_6^{(6)}$ defined similarly. Based on the number of times a generic triangulation is counted in $\sum_{i=1}^6 S_6^{(i)}$, find integers K and L with

$$K \cdot S_6 = L \cdot P_6,$$

and justify your answer.

2.11. For a regular n -gon, find integers K and L with

$$K \cdot S_n = L \cdot P_n,$$

where L indicates the number of times a fixed triangulation occurs in the count $\sum_{i=1}^n S_n^{(i)}$.

Lamé continues:

“But if one does the same for each of the other vertices of the polygon, and combines all the sums of the groups of these vertices, by their total sum $n(P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3)$ one will be certain to include all the partial decompositions of P_n ; each of these is itself repeated therein a certain number of times.

“Indeed, if one imagines an arbitrary such decomposition, it contains $n - 2$ triangles, having altogether $3n - 6$ sides; if one removes from this number the n sides of the polygon, and takes half of the remainder, which is $n - 3$, one will have the number of diagonals appearing in the given decomposition. Now, it is clear that this partial decomposition is repeated, in the preceding total sum, as many times as these $n - 3$ diagonals have ends, that is $2n - 6$ times: since each end is a vertex of the polygon, and in evaluating the groups of this vertex, the diagonal furnished a group including the particular partial decomposition under consideration.

“Thus, since each of the partial decompositions of the total group P_n is repeated $2n - 6$ times in $n(P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3)$, one obtains P_n upon dividing this sum by $2n - 6$. Therefore one has

$$(2) \quad P_n = \frac{n(P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3)}{2n - 6}.$$

2.12. In equation (2) identify terms which play the role of S_n , K and L .

Lamé’s equations (1) and (2) thus represent two strategies for computing P_n , although each equation is itself a recursion relation requiring the value of P_3, P_4, \dots, P_{n-1} to compute P_n . Can these two equations be combined to solve for P_n directly?

2.13. Find a fraction N with $P_{n+1} = NP_n$. Be sure to carefully justify your work. Use this equation to find a fraction N' with $P_{n+1} = N'P_{n-1}$.

Lamé concludes:

III.

“The first formula (1) gives

$$P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3 = P_{n+1} - 2P_n,$$

and the second (2) gives

$$P_3P_{n-1} + P_4P_{n-2} + \cdots + P_{n-2}P_4 + P_{n-1}P_3 = \frac{2n-6}{n}P_n;$$

so finally

$$P_{n+1} - 2P_n = \frac{2n-6}{n}P_n,$$

or

$$(3) \quad P_{n+1} = \frac{4n-6}{n}P_n.$$

This is what was to be proven."

Paris, 25 August, 1838.

3 A Modern Formula

3.1. Let $P_2 = 1$. Using the simple recursion relation $P_{n+1} = NP_n$, explain why

$$P_3 = \frac{2}{2}P_2$$

$$P_4 = \frac{2}{2} \cdot \frac{6}{3}P_2$$

Find an integer M with

$$P_5 = \frac{2 \cdot 6 \cdot M}{2 \cdot 3 \cdot 4}P_2.$$

Letting $M_1 = 2$ and $M_2 = 6$, find integers $M_3, M_4, M_5, \dots, M_{n-1}$ with

$$P_{n+1} = \frac{M_1 \cdot M_2 \cdot M_3 \cdot M_4 \cdots M_{n-1}}{2 \cdot 3 \cdot 4 \cdot 5 \cdots n}P_2.$$

We have $n! = 2 \cdot 3 \cdot 4 \cdot 5 \cdots n$. Factor a 2 from each M_i to write

$$P_{n+1} = 2^x \frac{(M_1/2)(M_2/2) \cdots (M_{n-1}/2)}{n!}P_2.$$

What is the value of x ? Do you recognize the product

$$\prod_{i=1}^{n-1} (M_i/2) = (M_1/2)(M_2/2)(M_3/2) \cdots (M_{n-1}/2)?$$

What is missing from $\prod_{i=1}^{n-1} (M_i/2)$ to form a factorial? Include the missing terms in both the numerator and denominator to write

$$P_{n+1} = \frac{(2n-2)!}{D_1 n!} P_2,$$

for some integer D_1 . Find an integer D_2 with

$$P_{n+1} = \frac{1}{D_2} \binom{2n-2}{n-1} P_2.$$

Justify your answer via a direct argument.

3.2. Use the above equation for P_{n+1} to compute P_{10} and compare this to exercise (2.3).

References

- [1] Euler, L., *Leonhard Euler und Christian Goldbach, Briefwechsel 1729–1764*, Juskevic, A. P., Winter, E. (eds.), Akademie Verlag, Berlin, 1965.
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- [4] Lamé, G., “Un polygone convexe étant donné, de combien de manières peut-on le partager en triangles au moyen de diagonales?,” *Journal de Mathématiques Pures et Appliquées*, **3** (1838), p. 505–507.